

On Coded Continuous Phase Modulation

Göran Lindell



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To my dear ones,

my father *Kurt*,
my mother *Margareta*,
my wife *Lena*,
and my daughters
Cecilia and *Catarina*

with my most sincere gratitude.

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ON CODED
CONTINUOUS PHASE MODULATION

SUMMARY

ABSTRACT

This thesis considers digital communication systems where combined convolutional coding and continuous phase modulation (CPM) are used.

The communication channel is assumed to be an additive white Gaussian noise channel. Furthermore, it is also assumed that the receiver is a coherent maximum likelihood receiver.

CPM is a constant amplitude digital modulation scheme with attractive spectral characteristics. By introducing convolutional coding the error probability is improved.

Applications, closely related to the study in this thesis, are satellite and space communication systems.

Power and bandwidth efficient digital modulation schemes are presented. By varying the system parameters, various combinations of power saving and bandwidth saving can be obtained.

Schemes with the same free Euclidean distance as the uncoded minimum shift keying (MSK) modulation scheme, but with only approximately half the bandwidth, are presented. Furthermore, schemes with roughly the same bandwidth as the uncoded MSK scheme, with a power gain in E_b/N_0 of about 4-6dB are also given.

An upper bound on the bit error probability is derived, and it is concluded that the free Euclidean distance is a very good performance measure for the considered class of signals.

The work presented in this thesis has been supported by SÖU, the National Swedish Board for Technical Development, under Grants 79-3594, 81-3480, 82-3254 and 83-3334.

I. INTRODUCTION

The purpose of a communication system is to communicate information from one place to another. Communication systems are used today in many applications. Examples are telephone, AM/FM radio, television, navigation, timing standards, mobile radio, radar landing systems, satellite and space communication.

There is an increased interest in digital communication systems. These systems are designed to communicate a sequence of random binary digits (bits, binary digit = bit). A binary digit can assume two values, 0 or 1, and these are typically represented as two different voltage levels.

There are many advantages with digital communication systems.

A very important advantage is that these systems can be used, without loss of optimality, to communicate any kind of information, even if the information is analog in nature (such as music). This is discussed in more detail by J.L. Massey in [1]. The underlying theory was published in 1948 by C.E. Shannon [2], the "father" of information theory.

Other advantages with digital communication systems are that coding techniques can be used to achieve reliable communication, and that privacy can be obtained by using encryption techniques. Furthermore, implementation aspects make digital communication systems attractive. Especially, with the rapid development of complex very large scale integrated (VLSI) circuits, advanced digital signal processing becomes inexpensive.

In many applications, the available transmitter power is a limited resource. Examples are satellite down-links and hand-held radio-telephones. Thus, the digital communication system should be designed to use its power in an efficient way.

The radio frequency band is also a limited resource. Therefore, it is advantageous if the transmitted signals are designed in such a way, that only a small portion of the frequency band is occupied. Furthermore, it is favourable if these signals also have a constant amplitude, since nonlinear amplifiers often are used.

The process of transforming a sequence of digital symbols to a transmitter signal, is referred to as a digital modulation scheme. In recent years many

such schemes have been studied. An example is continuous phase frequency shift keying (CPFSK) modulation [3]-[4]. A special case of the CPFSK modulation scheme is the minimum shift keying (MSK) modulation scheme [5]. See also [6] and references therein. These two schemes belong to an interesting class of digital modulation schemes, often referred to as the continuous phase modulation (CPM) class [7]-[9]. See also [10] and references therein. All schemes in this class use constant amplitude sinusoidal signals, and the phase of these signals carries the digital information. Furthermore, the phase is a continuous function of time and this yields attractive spectral properties. The continuity of the phase does also mean that memory is introduced in the signals, and this can be used by the receiver to improve the detection efficiency (power efficiency).

This thesis considers digital communication systems where combined convolutional coding and continuous phase modulation are used. Convolutional coding means that the information bit sequence is transformed (coded), by a convolutional encoder [11]-[13], to a new sequence of binary symbols. This new sequence is then fed into a CPM modulator. An important consequence of the use of convolutional coding, is that more memory is introduced in the signals.

The idea is to improve the error probability (power efficiency), while preserving the attractive spectral characteristics of the CPM modulation scheme. Our goal is to find schemes which are both power and bandwidth efficient.

Applications, closely related to the study in this thesis, are satellite and space communication systems.

Recently, several approaches to good combined channel coding and digital modulation schemes have been presented. Especially, the paper written by G. Ungerboeck [14], has received much attention. Examples of schemes studied in [14] are rate $2/3$ and rate $3/4$ convolutional codes, combined with 8 phase-shift keying (PSK) and 16 quadrature amplitude-shift keying (QASK) modulation, respectively. Rate $3/4$ convolutional codes combined with 16-PSK are considered in [15]. See also [16].

Independent work on convolutional coding combined with continuous phase modulation can be found in [17]. See also [18]-[27].

II. INTRODUCTION TO CONVOLUTIONAL CODING COMBINED WITH CONTINUOUS

PHASE MODULATION

The digital communication system studied in this thesis is described in detail in this part.

The communication channel is assumed to be an additive white Gaussian noise channel. Furthermore, the receiver is assumed to be a coherent maximum likelihood receiver [28], [12].

A trellis description [11]-[13], [29] of the transmitted signal is presented, which is valid for rational modulation indices.

It is also shown that the receiver can be implemented as a filter-bank combined with the Viterbi algorithm [11]-[13], [29]. This is the same structure as for uncoded CPM schemes [7]-[8], [30]-[31].

Asymptotically (large signal-to-noise ratios), the error probability is dominated by the free Euclidean distance [11]-[13], [28]-[29].

A simple way to upper bound the free Euclidean distance is described.

The estimated power spectral density [32]-[44], [23]-[24] and the estimated bandwidth (99% in band power), used in this thesis, are defined in this part.

Furthermore, the mainlobe and the asymptotic roll-off behaviour of the power spectral density, for some specific uncoded binary CPM schemes are studied.

The results on the power spectral density for uncoded binary CPM schemes, given in this part, are published in IEE Proceedings, Part F [46].

III. MINIMUM EUCLIDEAN DISTANCE FOR COMBINATIONS OF SHORT RATE 1/2

CONVOLUTIONAL CODES AND CPFSK MODULATION

In this part we study the Euclidean distance properties of signals formed by a rate 1/2 convolutional encoder followed by a binary or 4-level CPFSK modulator.

Especially, we have searched for those schemes that maximize the free Euclidean distance.

Independent works on coded CPM schemes are found in [17], [22] where also coded partial response CPM is studied. See also [26]-[27].

Symmetry properties are found, and these were used to reduce the number of convolutional encoders and mapping rules needed in the search.

The optimal codes are presented for the case of short rate 1/2 codes combined with binary CPFSK modulation.

For short rate 1/2 codes combined with 4-level CPFSK modulation, the optimal combination of code and mapper is presented.

Furthermore, the minimum Euclidean distance, as a function of the modulation index and the observation interval length is also studied in some detail.

It is found that the convolutional codes with optimum free Hamming distance [12] do not in general give the optimum free Euclidean distance with CPFSK. Schemes with a power gain in E_b/N_0 of about 5-6dB, compared to the uncoded MSK scheme, have been found.

Most of the work in this part is published in IEEE Transactions on Information Theory [47]. Parts of this work have been presented at two conferences; IEEE International Symposium on Information Theory 1982 [49] and IEEE Global Telecommunications Conference 1982 (GLOBECOM'82) [50].

IV. POWER AND BANDWIDTH EFFICIENT CODED MODULATION SCHEMES

In this part we study the asymptotic error probability (by means of the free Euclidean distance) of signals formed by combining high-rate convolutional codes with multi-level CPM.

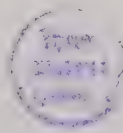
Independent work on rate $2/3$ convolutional encoders combined with 8-level CPFSK can be found in [21].

Code-independent and code-specific upper bounds on the free Euclidean distance have been identified for an interesting subclass of coded multi-level CPM schemes with a natural binary mapper.

We present schemes with the same asymptotic error probability as the uncoded MSK scheme, but with only approximately half the bandwidth (99% in band power). Schemes with roughly the same bandwidth as the uncoded MSK scheme, with a power gain in E_b/N_0 of about 4-6dB are also presented.

A method to reduce the spectral sidelobes while preserving the constant amplitude, the spectral mainlobe, and approximately the asymptotic error probability, is also introduced.

Most of the work in this part is published in Archiv für Elektronik und Übertragungstechnik [48]. Parts of this work have also been presented at three conferences; IEEE International Symposium on Information Theory 1983 [51], IEEE Global Telecommunications Conference 1983 (GLOBECOM'83) [52] and International Zürich Seminar on Digital Communications 1984 [53].



V. AN UPPER BOUND ON THE BIT ERROR PROBABILITY

In this part we study the bit error probability properties for the digital communication system considered in this thesis.

An upper bound on the bit error probability is derived, using the generating function technique [11]-[13]. This upper bound is valid for any choice of convolutional encoder, M-level mapper, frequency pulse and modulation index (rational).

Similar work can be found in [45] where the symbol error probability for uncoded CPM is considered.

For large signal-to-noise ratios the upper bound is dominated by the free Euclidean distance and by the characteristics of the signal pairs giving the free Euclidean distance.

The upper bound is evaluated numerically for a number of coded, as well as uncoded, multi-level full response CPFSK schemes.

We conclude that the free Euclidean distance is a very good performance measure for the considered class of signals.

Parts of this work will be presented at the IEEE International Symposium on Information Theory 1985 [54]. It is also to be submitted for publication in the Bell Laboratories Technical Journal [55].

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ON CODED
CONTINUOUS PHASE MODULATION

PART I

INTRODUCTION TO
CONVOLUTIONAL CODING COMBINED WITH
CONTINUOUS PHASE MODULATION

ABSTRACT

This thesis considers digital communication systems where combined convolutional coding and continuous phase modulation (CPM) are used. Continuous phase modulation is a constant amplitude digital modulation scheme with attractive spectral characteristics. Good error probability properties can be obtained with a coherent maximum likelihood receiver. An important consequence of the use of convolutional coding is that more memory is introduced in the signals. This can be used by the receiver to improve the error probability. Only coherent transmission over an additive white Gaussian noise channel is considered. Applications, closely related to the study in this thesis, are satellite and space communication systems.

The digital communication system is described in detail in this part. Furthermore, a trellis description of the transmitted signal, valid for rational modulation indices, is also given.

It is shown that the receiver can be implemented as a filter-bank combined with the Viterbi algorithm. This is the same structure as for uncoded CPM schemes.

Asymptotically (large E_b/N_0), the error probability is dominated by the free Euclidean distance. A simple way to upper bound the free Euclidean distance is described.

The estimated power spectral density and the estimated bandwidth (99% in band power), used in this thesis, are defined in this part. Furthermore, the mainlobe and the asymptotic roll-off behaviour of the power spectral density for some specific uncoded binary CPM schemes are studied.

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I. INTRODUCTION

The purpose of a communication system is to communicate information from one place to another. Communication systems are used today in many applications. Examples are telephone, AM/FM radio, television, navigation, timing standards, mobile radio, radar landing systems, satellite and space communication.

There is an increased interest in digital communication systems. These systems are designed to communicate a sequence of random binary digits (bits, binary digit = bit). A binary digit can assume two values, 0 or 1, and these are typically represented as two different voltage levels.

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Other advantages with digital communication systems are that coding techniques can be used to achieve reliable communication, and that privacy can be obtained by using encryption techniques. Furthermore, implementation aspects make digital communication systems attractive. Especially, with the rapid development of complex very large scale integrated (VLSI) circuits, advanced digital signal processing becomes inexpensive.

In many applications, the available transmitter power is a limited resource. Examples are satellite down-links and hand-held radio-telephones. Thus, the digital communication system should be designed to use its power in an efficient way.

The radio frequency band is also a limited resource. Therefore, it is advantageous if the transmitted signals are designed in such a way, that only a small portion of the frequency band is occupied. Furthermore, it is favourable if these signals also have a constant amplitude, since nonlinear amplifiers often are used.

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This thesis considers digital communication systems where combined convolutional coding and continuous phase modulation are used. Convolutional coding means that the information bit sequence is transformed (coded), by a convolutional encoder [11]-[13], to a new sequence of binary symbols. This new sequence is then fed into a CPM modulator. An important consequence of the use of convolutional coding, is that more memory is introduced in the signals.

The idea is to improve the error probability (power efficiency), while preserving the attractive spectral characteristics of the CPM modulation scheme. Our goal is to find schemes which are both power and bandwidth efficient.

Applications, closely related to the study in this thesis, are satellite and space communication systems.

Recently, several approaches to good combined channel coding and digital modulation schemes have been presented. Especially, the paper written by G. Ungerboeck [14], has received much attention. Examples of schemes studied in [14] are rate $2/3$ and rate $3/4$ convolutional codes, combined with 8 phase-shift keying (PSK) and 16 quadrature amplitude-shift keying (QASK) modulation, respectively. Rate $3/4$ convolutional codes combined with 16-PSK are considered in [15]. See also [16].

Independent work on convolutional coding combined with continuous phase modulation can be found in [17]. See also [18]-[27].

We will only consider transmission over an additive white Gaussian noise (AWGN) channel. Furthermore, it is assumed that the receiver is an ideal

coherent maximum likelihood receiver [28], [12]. Thus, it performs coherent maximum likelihood sequence detection (MLSD). For rational modulation indices this can be accomplished by using the Viterbi algorithm [11]-[13], [29].

The digital communication system studied in this thesis is described in detail in section II where the operations of the convolutional encoder, the mapper, the continuous phase modulator, the additive white Gaussian noise channel and the maximum likelihood receiver are explained. Furthermore, a trellis description [11]-[13], [29] of the transmitted signal, valid for rational modulation indices, is also given. In section III the error probability and the free Euclidean distance are considered, especially we focus our attention on a simple way to upper bound the free Euclidean distance.

In section IV, the estimated power spectral density [33]-[36] and the estimated bandwidth (99% in band power), used in this thesis, are defined. Furthermore, the mainlobe and the asymptotic roll-off behaviour of the power spectral density for some specific uncoded binary CPM schemes are studied. This part is terminated with a discussion and conclusion section.

II. SYSTEM DESCRIPTION

The digital communication system studied in this thesis is described in detail in this section.

We want to communicate a sequence of binary digits (bits). Denote this sequence by $\underline{u} = (\dots, u_{-1}, u_0, u_1, \dots)$, where $u_j \in \{0, 1\}$ all j . It is assumed that these information bits are generated at a constant rate of $1/T_b$ bits/second, equivalently a new bit is produced every T_b second.

The convolutional encoder:

The sequence $\underline{u}$ is encoded by a conventional convolutional encoder (k inputs and n outputs), [11]–[13], see fig. 1. The output from this encoder is a sequence of coded binary symbols which is denoted by $\underline{v} = (\dots, v_{-1}, v_0, v_1, \dots)$, where $v_j \in \{0, 1\}$, all j .

It is convenient to partition the sequence of information bits into blocks of length k , $\underline{u} = (\dots, \underline{u}_{-1}, \underline{u}_0, \underline{u}_1, \dots)$ where $\underline{u}_j = (u_{j,1}, u_{j,2}, \dots, u_{j,k}) = (u_{jk}, u_{jk+1}, \dots, u_{jk+k-1})$. In a similar way we also partition the sequence $\underline{v}$ into blocks of length n , $\underline{v} = (\dots, \underline{v}_{-1}, \underline{v}_0, \underline{v}_1, \dots)$ where $\underline{v}_j = (v_{j,1}, v_{j,2}, \dots, v_{j,n}) = (v_{jn}, v_{jn+1}, \dots, v_{jn+n-1})$. The convolutional encoder produces a block of coded symbols $\underline{v}_j$ each time a block of information bits $\underline{u}_j$ enters the encoder. The rate, R_c , of the convolutional encoder is therefore $R_c = k/n$ information bits/coded symbol. The encoding operation performed by the convolutional encoder can be written as

$$v_{j,m} = \sum_{\ell=1}^k \sum_{i=0}^{v_\ell} g_{m,\ell}(i) u_{j-i,\ell}; \quad m = 1, 2, \dots, n, \quad j = \dots, -1, 0, 1, \dots \quad (1)$$

where $g_{m,\ell}(i) \in \{0, 1\}$, the summations are modulo 2 and v_ℓ is the number of delay elements in the ℓ :th input. $g_{m,\ell}(0)$ represents the direct connection between the ℓ :th input and the m :th output. In general, $g_{m,\ell}(i)$ (if $i \geq 1$) represents the connection between the i :th delay element in input number ℓ , and the m :th output.

The total number of delay elements in the encoder is denoted by v , $v = v_1 + v_2 + \dots + v_k$. Furthermore, the state of the convolutional encoder is denoted by $\underline{x} = (x_1, x_2, \dots, x_v)$ where x_ℓ represents delay element number ℓ and $x_\ell \in \{0, 1\}$. Thus, the number of states of the convolutional encoder equals 2^v .

The mapping rule:

The coded sequence $\underline{v}$ is the input to a mapper. This mapper associates levels in an M -ary alphabet with blocks of coded symbols. The output from the mapper is a sequence $\underline{\alpha} = (\dots, \alpha_{-1}, \alpha_0, \alpha_1, \dots)$ of channel symbols, according to some mapping rule. It is assumed that M is a power of 2 and that $\alpha_j \in \{\pm 1, \pm 3, \dots, \pm(M-1)\}$, all j . Each time a block of coded symbols of length $\log_2(M)$ enters the mapper it produces one channel symbol. The rate, R_m , of the mapper is therefore $R_m = \log_2(M)$ coded symbols/channel symbol, and the overall rate, R , is $R = R_{c R_m}$ information bits/channel symbol.

The natural binary M -level mapping rule is defined by

$$\alpha_j = \sum_{i=0}^{s-1} v_{jR_m+i} \cdot 2^{(s-i)} - M+1, \quad j = \dots, -1, 0, 1, \dots \quad (2)$$

with $s = R_m$. The coded symbols v_{jR_m} and $v_{(j+1)R_m-1}$ are referred to as the most significant bit (MSB) and the least significant bit (LSB) respectively. The natural binary M -level mapping rule, eq. (2), is also given in table 1 for $M = 2, 4, 8, 16$. This table does also show some so-called Gray codes (Gray mappers). These mappers (Gray) are such that channel symbol levels which are close to each other correspond to inputs to the mapper which differ in only one position.

The combination of the convolutional encoder and the mapping rule can now be described in a convenient way. Let us first define k' and N' as: " k' equals the smallest number of information bits needed to produce an integer number ($=N'$) of channel symbols". Formally, k' is defined as the smallest positive integer such that (k'/k) equals an integer and such that $(k'/k)n = R_m N'$, where N' is a positive integer. It is seen that each time k' information bits enter the convolutional encoder, the mapper will produce N' channel symbols.

Examples: $k=1, n=2, M=2 \Rightarrow k'=k=1, N'=2$
 $M=2^n \Rightarrow k'=k, N'=1$
 $k=1, n=2, M=8 \Rightarrow k'=3, N'=2$
 $k=2, n=3, M=4 \Rightarrow k'=4, N'=3$
 uncoded CPM $\Rightarrow k'=R_m, N'=1$

Therefore we partition the sequence of information bits into blocks of length k' , $\underline{u} = (\dots, \underline{u}'_{-1}, \underline{u}'_0, \underline{u}'_1, \dots)$ where $\underline{u}'_j = (u'_{j,1}, u'_{j,2}, \dots, u'_{j,k'}) = (u_{jk'}, u_{jk'+1}, \dots, u_{jk'+k'-1})$. The state sequence of the convolutional encoder, associated with this partition of $\underline{u}$, is $\dots, \underline{x}_{-1}, \underline{x}_0, \underline{x}_1, \dots$ where $\underline{x}_j = (x_{j,1}, x_{j,2}, \dots, x_{j,v})$ is the state of the convolutional encoder given by the sequence $\dots, \underline{u}'_{j-2}, \underline{u}'_{j-1}$, see fig. 2.

In a similar way we also partition the channel symbol sequence into blocks of length N' , $\underline{\alpha} = (\dots, \underline{\alpha}'_{-1}, \underline{\alpha}'_0, \underline{\alpha}'_1, \dots)$ where

$$\underline{\alpha}'_j = (\alpha'_{j,1}, \alpha'_{j,2}, \dots, \alpha'_{j,N'}) = (\alpha_{jN'}, \alpha_{jN'+1}, \dots, \alpha_{jN'+N'-1}).$$

Then it is seen that $\underline{\alpha}'_j$ is completely known if the state $\underline{x}_j$ and if the input $\underline{u}'_j$ are known.

The continuous phase modulator (CPM):

The channel symbol sequence $\underline{\alpha}$ is the input to a CPM modulator, [7]-[10], [30]-[31]. This modulator produces the transmitted constant amplitude signal $s(t, \underline{\alpha})$,

$$s(t, \underline{\alpha}) = \sqrt{\frac{2E}{T}} \cos \left(2\pi f_0 t + \varphi(t, \underline{\alpha}) + \varphi_0 \right) \quad (3)$$

where the information carrying phase is

$$\varphi(t, \underline{\alpha}) = 2\pi h \sum_{i=-\infty}^{\infty} \alpha_i q(t - iT) \quad (4)$$

The channel symbol energy is denoted E , the channel symbol time T and the carrier frequency f_0 . h is the modulation index and φ_0 is an arbitrary constant phase shift. The transmitted energy per information bit, E_b , is $E_b = E/R$, and we also have $T_b = T/R$. The phase response $q(t)$ is defined by

$$q(t) = \int_{-\infty}^t g(\tau) d\tau \quad (5)$$

where $g(t)$ is the so called frequency pulse. It is assumed that $g(t)$ has finite length LT (L is a positive integer), i.e. $g(t) \equiv 0$ for $t < 0$ and for $t > LT$. Furthermore it is assumed that $q(LT) = 1/2$. If $L=1$ we have full response CPM and if $L > 1$ we have partial response CPM. Examples on classes of frequency pulses are the LREC class and the LRC class, see figs. 3-4. The expressions for $g(t)$ and $q(t)$, for different classes of frequency pulses, are given in appendix A.

Continuous phase frequency shift keying (CPFSK), [3]-[4], [10], is a special case of full response CPM. The frequency pulse is in this case

$$g(t) = \begin{cases} 1/2T & , 0 \leq t \leq T \\ 0 & , \text{otherwise} \end{cases} \quad (6)$$

If binary ($M=2$) CPFSK modulation is used together with the modulation index $h=1/2$ we have the well-known minimum shift keying (MSK) modulation scheme, [5]-[6].

State-description of the transmitted signal:

Let level j correspond to the time $t = jN'T$ and let interval j correspond to the time-interval $jN'T \leq t \leq (j+1)N'T$, see fig. 5.

The phase function $\varphi(t, \underline{\alpha})$ can, in interval j , be divided into two parts,

$$\varphi(t, \underline{\alpha}) = 2\pi h q(LT) \sum_{i=-\infty}^{jN'-L} \alpha_i + 2\pi h \sum_{i=jN'-L+1}^{(j+1)N'-1} \alpha_i q(t-iT) ; \quad jN'T \leq t \leq (j+1)N'T \quad (7)$$

The phase state at level j , θ_j , is defined by

$$\theta_j = \left(2\pi h q(LT) \sum_{i=-\infty}^{jN'-L} \alpha_i \right) \bmod 2\pi = \left(2\pi \frac{h}{p} \sum_{i=-\infty}^{jN'-L} \alpha_i \right) \bmod 2\pi \quad (8)$$

where it is assumed that the modulation index $h = 2\ell/p$ (ℓ and p integers with no common divisor) and that $q(LT) = 1/2$ (if $q(LT) = 0$ then $\theta_j = 0 \forall j$). Let S_θ denote the number of phase states.

Let us for a moment study the phase states for the uncoded CPM case ($N'=1$). If p is odd then $\theta_j \in \{0, 2\pi/p, 2 \cdot 2\pi/p, 3 \cdot 2\pi/p, \dots, (p-1) \cdot 2\pi/p\}$ all j . If p is even then $\dots, \theta_{j-2}, \theta_j, \theta_{j+2}, \dots$ belong to one set of phase states and $\dots, \theta_{j-1}, \theta_{j+1}, \theta_{j+3}, \dots$ belong to another set of phase states. Both sets have $p/2$ members. These two sets of phase states are $\{0, 2 \cdot 2\pi/p, 4 \cdot 2\pi/p, 6 \cdot 2\pi/p, \dots, (p-2) \cdot 2\pi/p\}$ and $\{2\pi/p, 3 \cdot 2\pi/p, 5 \cdot 2\pi/p, \dots, (p-1) \cdot 2\pi/p\}$. This is illustrated in fig. 6 for $p=8$, see also [10].

In general, for coded CPM, the number of phase states can never exceed the corresponding number of phase states for the uncoded CPM case. Therefore we have $S_0 \leq p$ if p is odd and $S_0 \leq p/2$ if p is even.

The second term in eq. (7) contains $L-1$ partial response channel symbols and the N' channel symbols $\underline{\alpha}'_j$. To be able to define the state associated with this term, at level j , we use the parameter η defined as the smallest nonnegative integer such that

$$\eta N' \geq L-1 \quad (9)$$

The parameter η equals the smallest number of blocks (of length N') needed to incorporate the $L-1$ partial response channel symbols. Now we define the state of the transmitted signal $s(t, \underline{\alpha})$ at level j , $\underline{\sigma}_j$, by

$$\underline{\sigma}_j = \begin{cases} (\theta_j, \underline{x}_{j-\eta}, \underline{u}'_{j-\eta}, \underline{u}'_{j-\eta+1}, \dots, \underline{u}'_{j-1}) & , \text{ if } L \geq 2 \\ (\theta_j, \underline{x}_j) & , \text{ if } L = 1 \end{cases} \quad (10)$$

It is seen that if the input to the convolutional encoder, $\underline{u}'_j$, is known and if the state of the transmitted signal at level j , $\underline{\sigma}_j$, is known, then the transmitted signal in interval j and the next state $\underline{\sigma}_{j+1}$ is known. The transmitted signal can therefore be described by a trellis [11]-[13], [29] with S states

$$S \leq S_0 \cdot 2^{(v+\eta k')} \leq p 2^{(v+\eta k')} \quad (11)$$

Furthermore, the number of transitions from each state equals $2^{k'}$. The first inequality in eq. (11) is necessary since all combinations of phase states and encoder states might not exist. Fig. 7a shows an example. The trellis for this scheme if $h=1/2$ is given in fig. 7b, and it is seen that for this modulation index $S_0=2=p/2$ and $S=S_0 \cdot 2=4$. The trellis for the scheme in fig. 7a with $h=2/3$ is given in fig. 7c, and it is seen that for this modulation index $S_0=2 < p=3$ and $S=2 < S_0 \cdot 2=4$.

The communication channel:

The communication channel is assumed to be an additive white Gaussian noise (AWGN) channel. Thus, the signal available for observation, $r(t)$, is $r(t) = s(t, \underline{\alpha}) + n(t)$ where $n(t)$ is a white Gaussian random process having zero mean and one-sided power spectral density N_0 .

The receiver:

The receiver is assumed to be an ideal coherent maximum likelihood receiver [28], [12]. Thus, it performs maximum likelihood sequence detection. The decision rule for this receiver can be formulated as: The receiver chooses the sequence $\hat{\underline{u}} \Leftrightarrow$ "the signal $s(t, \hat{\underline{\alpha}})$ is closer (in signal space) to $r(t)$ than any other possible transmitted signal". Now, since we have equal-energy signals this decision rule is equivalent with

$$J(\hat{\underline{u}}) = \max_{\{\underline{u}\}} J(\underline{u}) \quad (12)$$

where $J(\underline{u})$ is the correlation

$$J(\underline{u}) = \int r(t) s(t, \underline{\alpha}) dt \quad (13)$$

Eq. (13) can also be written (rational h)

$$J(\underline{u}) = \sum_j Z_j(\underline{u}'_j, \underline{\sigma}_j) \quad (14)$$

where the metric $Z_j(\underline{u}'_j, \underline{\sigma}_j)$ is given by

$$Z_j(\underline{u}'_j, \underline{\sigma}_j) = \frac{(j+1)N'T}{jN'T} \int r(t) s(t, \underline{\alpha}) dt \quad (15)$$

Then the decision rule in eq. (12) becomes

$$J(\hat{\underline{u}}) = \max_{\{\underline{u}\}} \sum_j Z_j(\underline{u}'_j, \underline{\sigma}_j) \quad (16)$$

Thus, the receiver shall find the path in the trellis which has the largest accumulated metric. A well-known procedure to find this path is the Viterbi algorithm [11]-[13], [29]. The number of metrics in interval j is $S \cdot 2^{k'}$. One way to obtain these metrics is to use the receiver structure in fig. 8. This is shown in appendix B.

III. ERROR PROBABILITY AND THE FREE EUCLIDEAN DISTANCE

The error probability, P_e , is a very important measure of quality. There are different kinds of error probabilities. Examples are the bit error probability, the symbol error probability and the error event probability [11]-[13], [28]-[29]. By using a union bound argument [28], the error probability can be upper bounded by a sum of terms [11]-[13], [28]-[29]. If the signal-to-noise ratio (SNR) E_b/N_0 is large then this sum, normally, is dominated by only one term. For large values on E_b/N_0 we therefore write

$$P_e \approx C \cdot Q\left(\sqrt{d_{\min}^2 E_b/N_0}\right) \quad (17)$$

where C is a constant independent of E_b/N_0 and where $Q(x)$ [28] is defined by

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-y^2/2} dy \quad (18)$$

$d_{\min}^2$ is referred to as the normalized squared **free** Euclidean distance and is defined by

$$d_{\min}^2 = \min_{\substack{\underline{u}, \underline{w} \\ \underline{u}_0' + \underline{w}_0' = 0}} \frac{1}{2E_b} \int_0^\infty [s(t, \underline{\alpha}) - s(t, \underline{\beta})]^2 dt \quad (19)$$

where $\underline{\alpha}$ and $\underline{\beta}$ are the output sequences from the mapper given by the input data sequences to the convolutional encoder $\underline{u}$ and $\underline{w}$ respectively.

The minimization in eq. (19) is carried out for all $\underline{u}$ and $\underline{w}$ such that

$$\underline{u}_m' = \underline{w}_m', \text{ for all } m \leq -1.$$

Eq. (17) motivates our study of $d_{\min}^2$. For comparison, $d_{\min}^2 = 2$ when uncoded MSK, uncoded binary phase shift keying (BPSK) or uncoded quaternary phase shift keying (QPSK) is used.

A more general parameter of interest is the minimum normalized squared Euclidean distance $d_{\min, N}^2$ defined by

$$d_{\min, N}^2 = \min_{\substack{\underline{u}, \underline{w} \\ \underline{u}_0' + \underline{w}_0' = 0}} \frac{1}{2E_b} \int_0^{NT} [s(t, \underline{\alpha}) - s(t, \underline{\beta})]^2 dt \quad (20)$$

with $\underline{u}$ and $\underline{w}$ the same as in eq. (19). $d_{\min, N}^2$ is the minimum normalized squared Euclidean distance between signals defined up to time NT as above.

It is a nondecreasing function of N with its maximum value equal to $d_{\min}^2$. It is interesting to know the value of N for which the free Euclidean distance is reached, i.e. the smallest N such that $d_{\min,N}^2 = d_{\min}^2$ because the path memory of the receiver (rational h) at large E_b/N_0 is related to this N [8]. A small N value is desirable. It is also of interest how fast $d_{\min,N}^2$ grows with N .

Upper bounds on the free Euclidean distance:

In eq. (19) it is always possible to find two information bit sequences $\underline{u}$ and $\underline{w}$ which differ in a finite number of positions and which are such that $s(t, \underline{\alpha}) = s(t, \underline{\beta})$ for $t \geq N_m T$ where N_m is some positive integer. The normalized squared Euclidean distance in signal space between these two signals is an upper bound, $d_B^2(h)$, on $d_{\min}^2(h)$.

$$d_B^2(h) = \frac{1}{2E_b} \int_0^{N_m T} [s(t, \underline{\alpha}) - s(t, \underline{\beta})]^2 dt \quad (21)$$

Clearly, $d_{\min,N}^2(h) \leq d_{\min}^2(h) \leq d_B^2(h)$. Consider the phase functions $\varphi(t, \underline{\alpha})$ and $\varphi(t, \underline{\beta})$ associated with the signals $s(t, \underline{\alpha})$ and $s(t, \underline{\beta})$ above. For $t \geq N_m T$ they are equal modulo 2π . This can be written as

$$h \cdot q(LT) \sum_{i=0}^{N_m-L} \gamma_i = n_1 \quad (22)$$

for some integer n_1 , and

$$\gamma_i = 0 \quad \text{for all } i \geq N_m - L + 1 \quad (23)$$

$\gamma_i = \alpha_i - \beta_i$ is the i :th element in the difference sequence $\underline{\gamma} = \underline{\alpha} - \underline{\beta}$ and $\gamma_i \in \{0, \pm 2, \pm 4, \dots, \pm 2(M-1)\}$. Eq. (22) represents a phase merger and eq. (23) represents a state merger in the convolutional encoder. The situation described above is therefore often referred to as a phase and state merger. Note that if $n_1 = 0$ in eq. (22) a phase and state merger has occurred independent of h while if $n_1 \neq 0$ a phase and state merger has oc-

curring only for specific h -values according to eq. (22). This is important because in the former case an upper bound on $d_{\min}^2(h)$ can be calculated for all h while in the latter case an upper bound can only be calculated for those specific h -values given by eq. (22). Such specific h -values are referred to as weak modulation indices.

To calculate $d_B^2(h)$ it is often convenient to use the function $d_{\text{inc}}^2(m, m_1)$ defined by

$$d_{\text{inc}}^2(m, m_1) = \frac{1}{2E_b} \int_{mT}^{(m+m_1)T} [s(t, \underline{\alpha}) - s(t, \underline{\beta})]^2 dt \quad (24)$$

which is the normalized squared Euclidean distance contribution over the time interval $mT \leq t \leq (m+m_1)T$. Assuming $f_0 T \gg 1$ straight-forward calculation yields

$$\begin{aligned} d_{\text{inc}}^2(m, m_1) &= R \cdot \frac{1}{T} \int_{mT}^{(m+m_1)T} \left(1 - \cos[\varphi(t, \underline{\alpha}) - \varphi(t, \underline{\beta})] \right) dt = \\ &= R \int_0^{m_1} \left(1 - \cos \left[2\pi h q(LT) \sum_{i=-\infty}^{m-L} \gamma_i + 2\pi h \sum_{i=0}^{m_1+L-2} \gamma_{m-L+1+i} q((x+L-1-i)T) \right] \right) dx \quad (25) \end{aligned}$$

When the frequency pulse $g(t)$ belongs to the LREC class of pulses further simplification is possible since the integral in eq. (25) can be solved analytically. The result is

$$d_{\text{inc}}^2(m, m_1) = R \left(m_1 - \sum_{j=0}^{m_1-1} C_j \right) \quad (26)$$

where

$$C_j = \begin{cases} \frac{\sin(y_j + z_j) - \sin(y_j)}{z_j} & , \quad z_j \neq 0 \\ \cos(y_j) & , \quad z_j = 0 \end{cases} \quad (27)$$

and where

$$z_j = \frac{\pi h}{L} \sum_{i=0}^{L-1} \gamma_{m+j-i} \quad (28)$$

and

$$y_j = \pi h \left(\sum_{i=-\infty}^{m+j-L} \gamma_i + \frac{1}{L} \sum_{i=0}^{L-1} i \gamma_{m+j-i} \right) \quad (29)$$

To produce a phase and state merger of length ℓ intervals ($N_m = \ell N'$), we must know the start-state of the convolutional encoder, x_0 , and the two information bit sequences $\underline{u}$ and $\underline{w}$. Furthermore, these sequences must have the following structure

$$\begin{cases} \underline{u} = (\underline{u}_0', \underline{u}_1', \dots, \underline{u}_{\ell-\eta-1}', \underline{u}_{\ell-\eta}', \dots, \underline{u}_{\ell-1}') \\ \underline{w} = (\underline{w}_0', \underline{w}_1', \dots, \underline{w}_{\ell-\eta-1}', \underline{u}_{\ell-\eta}', \dots, \underline{u}_{\ell-1}') ; \quad \underline{w}_0' \neq \underline{u}_0' \end{cases} \quad (30)$$

That is they have the same "tail", $\underline{u}_{\ell-\eta}', \dots, \underline{u}_{\ell-1}'$ (if $L \geq 2$). The sub-sequences $\underline{u}_0', \underline{u}_1', \dots, \underline{u}_{\ell-\eta-1}'$ and $\underline{w}_0', \underline{w}_1', \dots, \underline{w}_{\ell-\eta-1}'$ must both produce the same state of the convolutional encoder $x_{\ell-\eta}$. These sub-sequences must also be such that the corresponding channel symbol sub-sequences creates a phase merger (eq. (22)).

The normalized squared Euclidean distance, $d_B^2(h)$, for the phase and state merger given by eq. (30), can then be written as (see eq. (25))

$$d_B^2(h) = R \int_0^{N'} \left(1 - \cos \left[2\pi h \sum_{i=0}^{(\ell-\eta)N'-1} \gamma_i q((x-i)T) \right] \right) dx \quad (31)$$

Since the Euclidean distance for the phase and state merger only depends on $\gamma_0, \gamma_1, \dots, \gamma_{(\ell-\eta)N'-1}$, the "tail" can be replaced with zeroes without affecting $d_B^2(h)$. It is also seen that $\underline{u}$ and $\underline{w}$ in eq. (30) do not depend on the specific shape of the frequency pulse $g(t)$ (for fixed L). Furthermore, if $g(t)$ is replaced by a longer frequency pulse, then the same (in terms of $\underline{\gamma}$) phase and state merger can be generated by only adding enough zeroes to the "tails" in eq. (30). This is quite useful. Suppose we have found a phase and state merger for the full response case, then we can easily generate a phase and state merger for any other frequency pulse $g(t)$, and therefore be able to calculate an upper bound on the corresponding free Euclidean distance.

As an example of the above consider the modulation scheme in fig. 9a. This scheme consists of a rate $2/3$ convolutional encoder combined with 8-level (nat.bin.) CPFSK modulation (1REC). Now assume that the start state of the convolutional encoder equals $x_0 = (0,0,0)$. If the two sequences $\underline{u} = (00,11,00,01,00)$ and $\underline{w} = (01,00,10,01,00)$ are fed into the encoder, the corresponding channel symbol sequences are $\underline{\alpha} = (-7,5,-3,-1,-7)$ and $\underline{\beta} = (-3,-3,3,-7,-3)$ respectively. The phase functions $\varphi(t,\underline{\alpha})$ and $\varphi(t,\underline{\beta})$ are linearly increasing or linearly decreasing in each channel symbol interval depending on the corresponding channel symbol, see fig. 9b. It is seen that the two sequences $\underline{u}$ and $\underline{w}$ given above generate a phase and state merger, for all modulation indices h , after 5 ($=N_m$) channel symbol intervals, and the final state of the convolutional encoder is $x_5 = (0,1,0)$. Thus, an upper bound on $d_{\min}^2(h)$ for the specific coded CPFSK scheme given in fig. 9a can be calculated by using the $\underline{\gamma}$ -sequence $\underline{\gamma} = \underline{\alpha} - \underline{\beta} = (-4,8,-6,6,-4)$. Eqs. (26)-(29) with $m=0$, $m_1=5$, $R=2$ and $L=1$ yield

$$d_B^2(h) = 2 \left(5 - \frac{2}{3} \cdot \frac{\sin(2\pi h)}{2\pi h} - \frac{13}{3} \cdot \frac{\sin(4\pi h)}{4\pi h} \right) \quad (32)$$

By calculating $d_{\min}^2(h)$ for the scheme in fig. 9a, it is found that the upper bound is tight ($d_{\min}^2(h) = d_B^2(h)$) for modulation indices $h \leq 0.15$. More details on this scheme is given in Part III.

From [11]-[13], [32] we know that some convolutional encoders are referred to as catastrophic convolutional encoders. These encoders are associated with so-called catastrophic error propagation [11]-[13], [32]. In the same spirit we now define a catastrophic modulation scheme.

We say that the modulation scheme in fig. 1 is catastrophic if there exist two sequences $\underline{u}$ and $\underline{w}$ with infinite Hamming distance [12], such that the corresponding Euclidean distance is finite. These schemes are avoided to guarantee that catastrophic error propagation do not occur.

IV. POWER SPECTRAL DENSITY AND BANDWIDTH

The power spectral density of the transmitted signal is an important parameter to consider [33]-[36]. For the coded M-level CPM schemes studied in this thesis the normalized power spectral density is estimated by

$$\hat{G}_c(\beta) = R_c G(R_c \beta) \quad ; \quad \beta = f T_b \quad (33)$$

where $G(\beta)$ is the normalized power spectral density for the corresponding uncoded M-level CPM scheme [37]-[44]. See also [45] and references therein. Thus, the power spectrum is estimated to be expanded by a factor of $1/R_c$ compared to the uncoded case. In [23]-[24] P. Ho and P.J. McLane has calculated the true power spectral density for coded M-level CPM schemes. By comparing their results with our estimate $\hat{G}_c(\beta)$, it is concluded that our estimate is quite accurate, especially when high rate codes and multi-level CPM are used, see also [25].

The estimate $\hat{G}_c(\beta)$ is an equivalent baseband spectrum. The corresponding power spectral density at the carrier frequency f_0 is estimated by

$$\hat{P}_c(f) = \frac{P}{2} \left(\hat{S}_c(f-f_0) + \hat{S}_c(-f-f_0) \right) \quad (34)$$

where $P = E/T$ is the average transmitted signal power and where

$$\hat{S}_c(f) = T_b \hat{G}_c(f T_b) \quad (35)$$

This is illustrated in fig. 10.

A very useful function is the so-called fractional out of band power function $P_{ob}(\beta)$. This function reflects the amount of power contained outside the band $(-\beta, \beta)$, relative to the total power. For uncoded M-level CPM schemes, the fractional out of band power function is defined by

$$P_{ob}(\beta) = 10 \log_{10} \left(\frac{\int_{\beta}^{\infty} G(x) dx}{\int_0^{\infty} G(x) dx} \right) = 10 \log_{10} \left(2 \frac{\int_{\beta}^{\infty} G(x) dx}{\int_{-\beta}^{\beta} G(x) dx} \right) \quad (36)$$

To get an idea of how the power spectrum is affected by the choice of the frequency pulse $g(t)$ and by the choice of the modulation index h , let us study the fractional out of band power function for some specific uncoded binary CPM schemes, see figs. 11-14. All frequency pulses $g(t)$ are defined in appendix A. The power spectral densities, for the schemes considered in these figures, are derived from the general results given by Rowe and Prabhu [37] and Anderson and Salz [38]. Fig. 11 shows the fractional out of band power function when $g(t)$ is a triangular pulse of length $1T$ ($L=1$) and $2T$ ($L=2$) respectively. The modulation indices studied in this figure is $h=0.3$, $h=0.5$ and $h=0.7$. It is seen that the bandwidth increases with increasing modulation index, and decreases when partial response is used ($L=2$). It is also seen in this figure that when $\beta \gtrsim 3$ the fractional out of band power functions decrease approximately with the same rate. An expression on the power spectral density for the 2TRI case is given in appendix C. This expression contains a Fresnel integral which is evaluated by using standard computer programs [46].

Baker [39] has shown that for frequencies very far from the carrier frequency, the power spectral density decreases asymptotically as $\sim f^{-(2m+6)}$, where m is the number of continuous derivatives of the pulse $g(t)$. This is illustrated in figs. 12-13 for pulses with an increasing number of continuous derivatives. The modulation index is in these figures $h=1/2$. The pulses Δ REC, REC, TRI, RC and CRC corresponds to m -values $-2, -1, 0, 1$ and 2 respectively. The Δ REC pulses are very special since they have a discontinuous phase response. These pulses (Δ REC) are used here only to illustrate the asymptotic decay. It is seen in figs. 12-14 that pulses with more continuous derivatives yield a power spectral density with a wider mainlobe and with a faster asymptotic roll-off. Note the similarities in the asymptotic region between REC and RC and also between TRI and CRC. In appendix C expressions on the power spectral density are given for the 2RC case and for the 2CRC case respectively.

Recently, T. Aulin and C-E. Sundberg found an easy and very fast method to calculate the power spectral density for uncoded M -level CPM schemes [45]. By using this method it is actually no problem to calculate the estimate $\hat{G}_c(\beta)$ eq. (33).

Bandwidth:

The definition of bandwidth used in this thesis is the 99% in band power definition. Equivalently, the -20dB level in the fractional out of band power function can be used. The estimated normalized double-sided bandwidth $2\hat{B}_c T_b$ is

$$2\hat{B}_c T_b = 2BT_b/R_c \quad (37)$$

where $2BT_b$ is the normalized double-sided bandwidth (99% in band power) for the corresponding uncoded M-level CPM scheme. As a reference, the uncoded MSK scheme has $2BT_b = 1.18$. Fig. 15 shows the bandwidth $2BT_b$ as a function of the modulation index h for some uncoded M-level CPFSK schemes, $M = 2, 4, 8$ and 16 . This figure does also show the normalized double-sided bandwidth if the 99.9% in band power definition is used (the -30dB level in the fractional out of band power function). It is seen in this figure that the bandwidth depends on the modulation index in a non-trivial way. In table 2, the bandwidth is given for some uncoded M-level CPFSK schemes, $M = 2, 4, 8, 16$ and 32 . This table will be used frequently in this thesis when coded M-level CPFSK schemes are studied.

V. DISCUSSION AND CONCLUSIONS

The digital communication system studied in this thesis consists of a rate k/n convolutional encoder, an M -level mapper, a CPM-modulator, an AWGN channel and a coherent maximum likelihood receiver.

This system is described in detail in this part. Furthermore, a trellis description of the transmitted signal, valid for rational modulation indices, is presented. It is also shown that the receiver can be implemented as a filter-bank combined with the Viterbi algorithm. This is the same structure as for uncoded CPM schemes [7]-[8], [30]-[31].

Asymptotically (large E_b/N_0), the error probability is dominated by the free Euclidean distance. A simple way to upper bound the free Euclidean distance is described.

The bandwidth definition used in this thesis is the double-sided frequency band containing 99% of the average transmitted signal power, normalized with the information bit rate $1/T_b$. We estimate this bandwidth by $2\hat{B}_c T_b = 2BT_b/R_c$ where $2BT_b$ is the normalized double-sided bandwidth for the uncoded M -level CPM scheme, and where R_c is the rate of the convolutional encoder ($R_c = k/n$). By comparing the estimated bandwidth with the true bandwidth in [23]-[24] it is concluded that the estimated bandwidth is a good estimate.

The mainlobe and the asymptotic roll-off behaviour of the power spectral density for some specific uncoded binary CPM schemes are also studied. Especially, schemes with frequency pulses with an increasing number of continuous derivatives are considered.

APPENDIX A: SOME CLASSES OF FREQUENCY PULSES AND THEIR PHASE RESPONSE

The class of rectangular frequency pulses, LREC:

$$g(t) = \begin{cases} 1/2LT, & 0 \leq t \leq LT \\ 0, & \text{otherwise} \end{cases} \quad (A1)$$

$$q(t) = \begin{cases} 0, & t \leq 0 \\ t/2LT, & 0 \leq t \leq LT \\ 1/2, & t \geq LT \end{cases} \quad (A2)$$

See also figs. 3-4.

The class of half cycle sinus frequency pulses, LHCS:

$$g(t) = \begin{cases} \pi \sin(\pi t/LT)/4LT, & 0 \leq t \leq LT \\ 0, & \text{otherwise} \end{cases} \quad (A3)$$

$$q(t) = \begin{cases} 0, & t \leq 0 \\ (1 - \cos(\pi t/LT))/4, & 0 \leq t \leq LT \\ 1/2, & t \geq LT \end{cases} \quad (A4)$$

See also figs. 3-4.

The class of triangular frequency pulses, LTRI:

$$g(t) = \begin{cases} 2t/(LT)^2, & 0 \leq t \leq LT/2 \\ 2(1-t/LT)/LT, & LT/2 \leq t \leq LT \\ 0, & \text{otherwise} \end{cases} \quad (A5)$$

$$q(t) = \begin{cases} 0, & t \leq 0 \\ (t/LT)^2, & 0 \leq t \leq LT/2 \\ 2t/LT - (t/LT)^2 - 1/2, & LT/2 \leq t \leq LT \\ 1/2, & t \geq LT \end{cases} \quad (A6)$$

See also figs. 3-4.

The class of raised cosine frequency pulses, LRC:

$$g(t) = \begin{cases} (1 - \cos(2\pi t/LT))/2LT, & 0 \leq t \leq LT \\ 0, & \text{otherwise} \end{cases} \quad (A7)$$

$$q(t) = \begin{cases} 0, & t \leq 0 \\ t/2LT - \sin(2\pi t/LT)/4\pi, & 0 \leq t \leq LT \\ 1/2, & t \geq LT \end{cases} \quad (A8)$$

See also figs. 3-4.

The LAREC frequency pulse:

$$g(t) = \begin{cases} \frac{\Delta}{2} \delta(t) + \frac{1/2-\Delta}{LT} + \frac{\Delta}{2} \delta(t-LT), & 0 \leq t \leq LT \\ 0, & \text{otherwise} \end{cases} \quad (A9)$$

with

$$0 \leq \Delta \leq 1/2 \quad (A10)$$

The convolved raised cosine frequency pulse 1CRC:

$$g(t) = \begin{cases} \frac{2t}{T} - \frac{1}{2\pi T} \sin(4\pi t/T), & 0 \leq t \leq T/2 \\ \frac{2}{T} - \frac{2t}{T^2} + \frac{1}{2\pi T} \sin(4\pi t/T), & T/2 \leq t \leq T \\ 0, & \text{otherwise} \end{cases} \quad (A11)$$

This pulse is the convolution between " $\frac{1}{2}$ REC" and " $\frac{1}{2}$ RC", normalized such that $q(T) = 1/2$.

The convolved raised cosine frequency pulse 2CRC:

$$g(t) = \begin{cases} \frac{t}{2T^2} - \frac{1}{4\pi T} \sin(2\pi t/T) & , \quad 0 \leq t \leq T \\ \frac{1}{T} - \frac{t}{2T^2} + \frac{1}{4\pi T} \sin(2\pi t/T) & , \quad T \leq t \leq 2T \\ 0 & , \quad \text{otherwise} \end{cases} \quad (A12)$$

This pulse is the convolution between 1REC and 1RC, normalized such that $q(2T) = 1/2$.

The class of "1REC-2RC" frequency pulses:

$$g(t) = \begin{cases} (1 - \cos(\pi t/\epsilon T))/4T & , \quad 0 < t < \epsilon T \\ 1/2T & , \quad \epsilon T \leq t \leq T \\ (1 + \cos(\pi(t-T)/\epsilon T))/4T & , \quad T < t < (1+\epsilon)T \\ 0 & , \quad \text{otherwise} \end{cases} ; \quad 0 \leq \epsilon \leq 1 \quad (A13)$$

$$q(t) = \begin{cases} 0 & , \quad t \leq 0 \\ t/4T - \epsilon \sin(\pi t/\epsilon T)/4\pi & , \quad 0 < t < \epsilon T \\ t/2T - \epsilon/4 & , \quad \epsilon T \leq t \leq T \\ t/4T + (1-\epsilon)/4 + \epsilon \sin(\pi(t-T)/\epsilon T)/4\pi & , \quad T < t < (1+\epsilon)T \\ 1/2 & , \quad t \geq (1+\epsilon)T \end{cases} ; \quad 0 \leq \epsilon \leq 1 \quad (A14)$$

See also figs. 2-3 in Part III. The frequency pulse eq. (A13) is obtained by convolving the 1REC frequency pulse with the function $c(t)$

$$c(t) = \begin{cases} \pi \sin(\pi t/\epsilon T)/2\epsilon T & , \quad 0 \leq t \leq \epsilon T \\ 0 & , \quad \text{otherwise} \end{cases} ; \quad 0 < \epsilon \leq 1 \quad (A15)$$

APPENDIX B: A RECEIVER STRUCTURE

The metric for a specific transition (in the trellis) in interval j is

$$Z_j(\underline{u}_j', \underline{\sigma}_j) = \int_{jN'T}^{(j+1)N'T} r(t) s(t, \underline{\alpha}) dt \quad (B1)$$

Let us now write $s(t, \underline{\alpha})$ in eq. (B1) as

$s(t, \underline{\alpha}) = \sqrt{2E/T} \cos(\omega_0 t + \theta_j + \tilde{\varphi}(t, \underline{\alpha}) + \varphi_0)$ (in interval j). $\tilde{\varphi}(t, \underline{\alpha})$ is given by eq. (7)

$$\tilde{\varphi}(t, \underline{\alpha}) = 2\pi h \sum_{i=jN'-L+1}^{(j+1)N'-1} \alpha_i q(t-iT) \quad (B2)$$

The $L-1+N'$ channel symbols which determines $\tilde{\varphi}(t, \underline{\alpha})$ are completely known since $\underline{u}_j'$ and $\underline{\sigma}_j$ are known. By using trigonometric identities eq. (B1) can be written as

$$\begin{aligned} \sqrt{T/2E} Z_j(\underline{u}_j', \underline{\sigma}_j) &= \cos(\theta_j) \int_{jN'T}^{(j+1)N'T} r(t) \cos(\omega_0 t + \varphi_0) \cos(\tilde{\varphi}(t, \underline{\alpha})) dt - \\ &- \sin(\theta_j) \int_{jN'T}^{(j+1)N'T} r(t) \cos(\omega_0 t + \varphi_0) \sin(\tilde{\varphi}(t, \underline{\alpha})) dt - \\ &- \sin(\theta_j) \int_{jN'T}^{(j+1)N'T} r(t) \sin(\omega_0 t + \varphi_0) \cos(\tilde{\varphi}(t, \underline{\alpha})) dt - \\ &- \cos(\theta_j) \int_{jN'T}^{(j+1)N'T} r(t) \sin(\omega_0 t + \varphi_0) \sin(\tilde{\varphi}(t, \underline{\alpha})) dt \end{aligned} \quad (B3)$$

Define the impulse response $f_c(t)$ and the impulse response $f_s(t)$ as

$$f_c(t) = \begin{cases} \cos\left(2\pi h \sum_{i=-L+1}^{N'-1} \tilde{\alpha}_i q((N'-i)T-t)\right) & , \quad 0 \leq t \leq N'T \\ 0 & , \quad \text{elsewhere} \end{cases} \quad (B4)$$

$$f_s(t) = \begin{cases} \sin\left(2\pi h \sum_{i=-L+1}^{N'-1} \tilde{\alpha}_i q((N'-i)T-t)\right) & , \quad 0 \leq t \leq N'T \\ 0 & , \quad \text{elsewhere} \end{cases} \quad (B5)$$

where $\tilde{\alpha}_{-L+1}, \dots, \tilde{\alpha}_{N'-1}$ is the same sequence of channel symbols as in eq. (B2). It is seen that $f_c(t)$ and $f_s(t)$ are matched filters. Eq. (B3) indicates that the structure given in fig. B1 can be used to obtain the metric $Z_j(\underline{u}_j', \underline{\sigma}_j)$.

The bandpass filter in fig. B1 is centered at f_0 and its pass-band is large enough to let the signal-part in $r(t)$ pass the filter essentially undistorted. The lowpass filters in fig. B1 both have a bandwidth which is larger than the bandwidth of the filters $f_c(t)$ and $f_s(t)$. The low-pass filters remove the frequency-components around the double carrier-frequency $2f_0$. Also in this figure, x_j and y_j are created from the 4 samples taken at time $t=jN'T$. The metric $Z_j(u'_j, \sigma_j)$ is now given by

$$\sqrt{\frac{T}{2E}} Z_j(u'_j, \sigma_j) = \cos(\theta_j) x_{j+1} + \sin(\theta_j) y_{j+1} \quad (B6)$$

Note that $S_\theta - 1$ additional metrics (in interval j) can be calculated by letting θ_j in eq. (B6) take $S_\theta - 1$ additional values. Thus, the metrics for S_θ specific transitions in the trellis (in interval j) can be obtained by the implementation given in fig. B1. The corresponding S_θ metrics in interval $j+1$ are calculated in the same way from the values x_{j+2} and y_{j+2} . Thus, the metrics are obtained in a sequential manner.

To get all metrics in interval j we must have $2^{k'} \cdot 2^{v+\eta k'}$ filter-banks where each filter-bank has 4 filters as in fig. B1. However, the filters are completely specified by the corresponding sequence of $L-1+N'$ channel symbols $\tilde{\alpha}_{-L+1}, \dots, \tilde{\alpha}_{N'-1}$. Since there are $M^{L-1+N'}$ such sequences it is enough to have $M^{L-1+N'}$ filter-banks to be able to obtain all metrics in interval j . A complete receiver structure is given in fig. 8.

APPENDIX C: THE POWER SPECTRAL DENSITY FOR UNCODED BINARY 2TRI, 2RC AND 2CRC SCHEMES

By specializing the general result on the power spectral density, given by Rowe and Prabhu in [37], to binary CPM schemes with $L=2$, $q(LT) = 1/2$ and with a noninteger modulation index h , $S(f)$ can be written as

$$(P(f) = \frac{P}{2} (S(-f-f_0) + S(f-f_0)))$$

$$\begin{aligned} S(f) = & \frac{1}{T} \left[K_1(f) |R_1(f)|^2 + K_1(-f) |R_1(-f)|^2 + K_2(f) (|R_2(f)|^2 + |R_2(-f)|^2) + \right. \\ & + 2\text{Re} \left\{ K_3(f) R_1(f) R_2^*(f) + K_3(-f) R_1(-f) R_2^*(-f) + K_4(f) R_1(f) R_2(-f) + \right. \\ & \left. \left. + K_4(-f) R_1(-f) R_2(f) + K_5(f) R_1(f) R_1(-f) + K_6(f) R_2(f) R_2(-f) \right\} \right] \quad (C1) \end{aligned}$$

where $(\omega = 2\pi f, j = \sqrt{-1}, \text{Re}\{(\cdot)\}$ denotes real part of $(\cdot)$)

$$R_1(f) = \int_0^T \exp \left\{ j \left(-V(t) - V(t-T) \right) \right\} \cdot \exp \{-j\omega t\} dt \quad (C2)$$

$$R_2(f) = \int_0^T \exp \left\{ j \left(-V(t) + V(t-T) \right) \right\} \cdot \exp \{-j\omega t\} dt \quad (C3)$$

and where

$$V(t) = 2\pi h \int_0^{t+T} g(\tau) d\tau \quad (C4)$$

The frequency pulses 2TRI, 2RC and 2CRC are all such that $g(t) + g(t+T) = 1/2T$ for $0 \leq t \leq T$. Therefore, $R_1(f)$ is the same for these pulses,

$$R_1(f) = T \frac{\sin(\omega T/2 + h\pi/2)}{\omega T/2 + h\pi/2} \cdot \exp \{-j(\omega T/2 + h\pi)\} \quad (C5)$$

The functions $K_1(f), \dots, K_6(f)$ do not depend on the specific shape of the frequency pulse. If the functions $c_{k,l} = \cos(k\omega T + l\pi h)$ and $s_{k,l} = \sin(k\omega T + l\pi h)$, are introduced then

$$K_1(f) = (1+c_{1,1})/4+(c_{2,2}-Cc_{1,2})/8N(f) \quad (C6)$$

$$K_2(f) = 1/4+(c_{2,0}-Cc_{1,0})/8N(f) \quad (C7)$$

$$K_3(f) = c_{1,1}/8+(c_{2,0}-Cc_{1,0}+c_{2,2}-Cc_{1,2})/16N(f) + \\ + j\left(s_{1,1}/8+(Cs_{1,0}-s_{2,0}+s_{2,2}-Cs_{1,2})/16N(f)\right) \quad (C8)$$

$$K_4(f) = c_{1,-1}/8+(c_{2,0}-Cc_{1,0}+c_{2,2}-Cc_{1,2})/16N(f) + \\ + j\left(-s_{1,-1}/8+(Cs_{1,0}-s_{2,0}+s_{2,2}-Cs_{1,2})/16N(f)\right) \quad (C9)$$

$$K_5(f) = \left(c_{2,-2}-Cc_{1,-2}+c_{2,2}-Cc_{1,2}+j(Cs_{1,-2}-s_{2,-2}+s_{2,2}-Cs_{1,2})\right)/16N(f) \quad (C10)$$

$$K_6(f) = (c_{1,-1}+c_{1,1})/8+(c_{2,0}-Cc_{1,0})/8N(f)+j(s_{1,1}-s_{1,-1})/8 \quad (C11)$$

where $N(f) = 1+C^2-2C\cos(\omega T)$ and $C = \cos(\pi h)$.

To be able to calculate $S(f)$ we must also know the function $R_2(f)$, which now is given for the 2TRI, 2RC and 2CRC case respectively.

Triangle case (2TRI):

$$R_2(f) = T \exp\left\{-j\left(h\pi/2 + \frac{(\omega T+h\pi)^2}{4h\pi}\right)\right\} \cdot \frac{I(f)}{\sqrt{2h}} \quad (C12)$$

where

$$I(f) = \int_{x(f)}^{x(f)+\sqrt{2h}} \exp\left\{j \frac{\pi x^2}{2}\right\} dx \quad (C13)$$

and

$$x(f) = -\frac{\omega T+h\pi}{\sqrt{2h} \pi} \quad (C14)$$

$I(f)$ is calculated by using standard computer programs for Fresnel integrals, [46].

Raised cosine case (2RC):

$$R_2(f) = T \exp\left\{-j \frac{\omega T + h\pi}{2}\right\} \cdot \sum_{n=-\infty}^{\infty} J_n(h) \frac{\sin(\omega T/2 + n\pi/2)}{\omega T/2 + n\pi/2} \cdot \exp\left\{-j \frac{n\pi}{2}\right\} \quad (C15)$$

where $J_n(x)$ is the Bessel-function of the first kind and of order n .
A simple way to calculate $J_n(x)$ is described in [46].

Convolved raised cosine case (2CRC):

$$R_2(f) = T \exp\left\{-j \left(\frac{h}{2\pi} + \frac{h\pi}{2} + \frac{(\omega T + h\pi)^2}{4h\pi} \right)\right\} \cdot \sum_{n=-\infty}^{\infty} G(n) \exp\left\{-j \frac{n^2 \pi^2 - nh\pi^2 - n\pi\omega T}{h\pi}\right\} \cdot \frac{I(n, f)}{\sqrt{2h}} \quad (C16)$$

where $G(n) = G(-n)$ and where

$$G(n) = (-1)^{\frac{n}{2}} J_n\left(\frac{h}{2\pi}\right), \quad n = 0, 2, 4, 6, 8, \dots \quad (C17)$$

$$G(n) = j(-1)^{\frac{n-1}{2}} J_n\left(\frac{h}{2\pi}\right), \quad n = 1, 3, 5, 7, 9, \dots \quad (C18)$$

$$I(n, f) = \int_{x(n, f)}^{x(n, f) + \sqrt{2h}} \exp\left\{j \frac{\pi x^2}{2}\right\} dx \quad (C19)$$

with

$$x(n, f) = - \frac{\omega T - n2\pi + h\pi}{\sqrt{2h} \pi} \quad (C20)$$

Input to the mapper	Output channel symbol
1	1
0	-1

a) M=2

Input to the mapper		Output channel symbol	
MSB	LSB	Natural binary	Gray mapper
1	1	3	1
1	0	1	3
0	1	-1	-1
0	0	-3	-3

b) M=4

Input to the mapper			Output channel symbol	
MSB		LSB	Natural binary	Gray mapper
1	1	1	7	5
1	1	0	5	7
1	0	1	3	3
1	0	0	1	1
0	1	1	-1	-5
0	1	0	-3	-7
0	0	1	-5	-3
0	0	0	-7	-1

c) M=8

Input to the mapper				Output channel symbol	
MSB			LSB	Natural binary	Gray mapper
1	1	1	1	15	5
1	1	1	0	13	7
1	1	0	1	11	3
1	1	0	0	9	1
1	0	1	1	7	11
1	0	1	0	5	9
1	0	0	1	3	13
1	0	0	0	1	15
0	1	1	1	-1	-5
0	1	1	0	-3	-7
0	1	0	1	-5	-3
0	1	0	0	-7	-1
0	0	1	1	-9	-11
0	0	1	0	-11	-9
0	0	0	1	-13	-13
0	0	0	0	-15	-15

d) M=16

Table 1: M-level mapping rules.

a) M=2

b) M=4

c) M=8

d) M=16

The normalized double-sided bandwidth $2B_{\text{b}}$ when uncoded M-level CPFSK modulation is used.

h	99% in band power (-20dB)					99.9% in band power (-30dB)				
	M=2	M=4	M=8	M=16	M=32	M=2	M=4	M=8	M=16	M=32
0.05	0.184	0.246	0.288	0.328	0.456	0.654	0.524	0.476	0.582	0.824
0.06	0.238	0.292	0.322	0.390	0.528	0.744	0.568	0.552	0.678	0.928
0.07	0.292	0.332	0.352	0.450	0.592	0.820	0.606	0.656	0.794	1.024
0.08	0.344	0.366	0.378	0.492	0.656	0.886	0.640	0.692	0.914	1.112
0.09	0.394	0.398	0.404	0.530	0.720	0.942	0.670	0.728	0.976	1.216
0.10	0.442	0.426	0.434	0.568	0.784	0.992	0.702	0.772	1.028	1.304
0.11	0.486	0.452	0.468	0.608	0.848	1.038	0.738	0.826	1.086	1.408
0.12	0.528	0.474	0.514	0.654	0.912	1.078	0.790	0.898	1.158	1.512
0.13	0.568	0.496	0.560	0.698	0.976	1.114	0.920	0.970	1.224	1.600
0.14	0.604	0.518	0.596	0.740	1.040	1.148	0.972	1.052	1.276	1.704
0.15	0.640	0.536	0.624	0.778	1.096	1.180	1.000	1.150	1.328	1.784
0.16	0.672	0.556	0.652	0.814	1.160	1.208	1.026	1.216	1.382	1.872
0.17	0.702	0.574	0.676	0.854	1.224	1.234	1.050	1.260	1.448	1.952
0.18	0.732	0.592	0.702	0.896	1.288	1.260	1.076	1.298	1.514	2.032
0.19	0.760	0.610	0.726	0.938	1.352	1.284	1.104	1.332	1.570	2.112
0.20	0.786	0.630	0.750	0.978	1.416	1.308	1.136	1.366	1.624	2.192
0.21	0.810	0.650	0.774	1.016	1.472	1.330	1.170	1.402	1.684	2.264
0.22	0.834	0.674	0.802	1.054	1.536	1.352	1.210	1.440	1.756	2.352
0.23	0.856	0.702	0.830	1.092	1.600	1.374	1.260	1.486	1.828	2.424
0.24	0.878	0.734	0.862	1.132	1.664	1.396	1.320	1.536	1.888	2.496
0.25	0.898	0.772	0.892	1.172	1.720	1.420	1.378	1.582	1.944	2.576
0.26	0.918	0.806	0.920			1.446	1.430	1.624		
0.27	0.936	0.838	0.948			1.478	1.482	1.660		
0.28	0.952	0.864	0.974			1.518	1.536	1.694		
0.29	0.970	0.888	1.000			1.606	1.602	1.726		
0.30	0.986	0.910	1.024			1.874	1.680	1.758		
0.31	1.000	0.930	1.048			1.918	1.746	1.792		
0.32	1.014	0.950	1.072			1.948	1.794	1.828		
0.33	1.028	0.968	1.096			1.972	1.834	1.868		
0.34	1.042	0.984	1.122			1.994	1.866	1.912		
0.35	1.054	1.002	1.148			2.016	1.896	1.958		
0.36	1.066	1.018	1.176			2.036	1.922	2.000		
0.37	1.076	1.034	1.204			2.058	1.948	2.040		
0.38	1.086	1.050	1.230			2.078	1.972	2.076		
0.39	1.098	1.066	1.258			2.100	1.994	2.112		
0.40	1.106	1.082	1.284			2.122	2.018	2.146		
0.41	1.116	1.098	1.308			2.146	2.040	2.182		
0.42	1.124	1.114	1.334			2.172	2.062	2.222		
0.43	1.132	1.130	1.358			2.198	2.086	2.268		
0.44	1.140	1.146	1.380			2.228	2.110	2.322		
0.45	1.148	1.164	1.404			2.262	2.136	2.378		
0.46	1.156	1.182	1.428			2.304	2.164	2.428		
0.47	1.162	1.202	1.454			2.376	2.196	2.474		
0.48	1.170	1.222	1.480			2.628	2.232	2.514		
0.49	1.176	1.244	1.506			2.690	2.272	2.550		
0.50	1.182	1.268	1.532			2.736	2.310	2.586		

Table 2. The normalized double-sided bandwidth, $2B_{\text{b}}$, when uncoded M-level CPFSK modulation is used. The levels in the fractional out of band power function used in the two definitions of bandwidth are -20[dB] and -30[dB]. See also fig. 15.

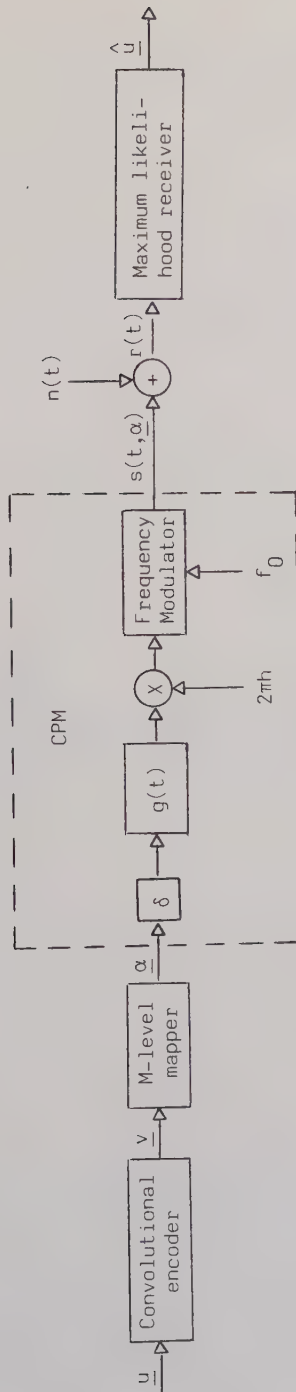


Figure 1. The modulation scheme, channel and receiver.

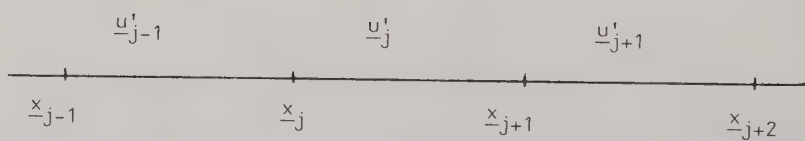


Figure 2. Illustrating the partitioning of the information bits into blocks of length k' and the associated state sequence of the convolutional encoder.

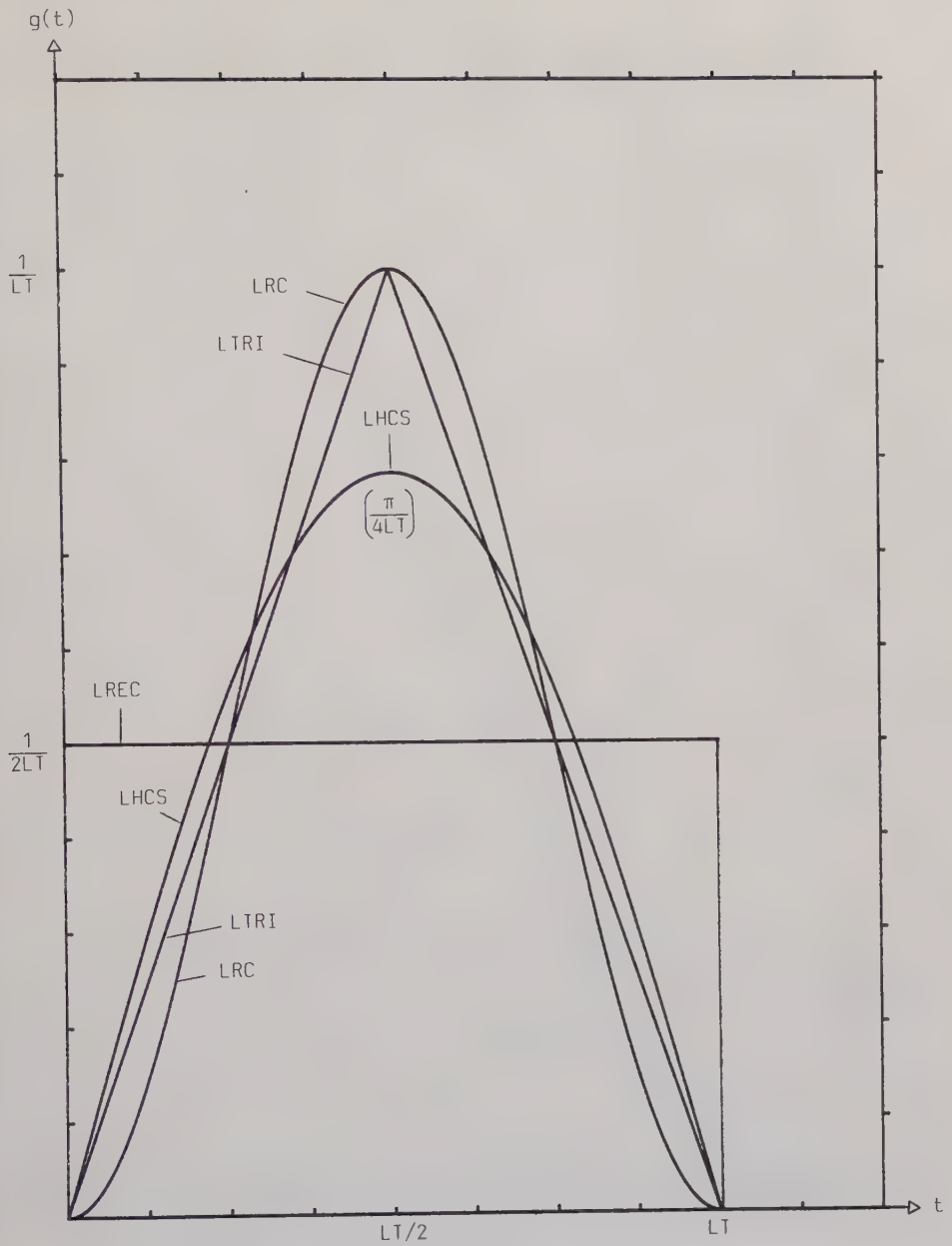


Figure 3. The LREC, LHCS, LTRI and LRC class of frequency-pulses $g(t)$.
See also appendix A.

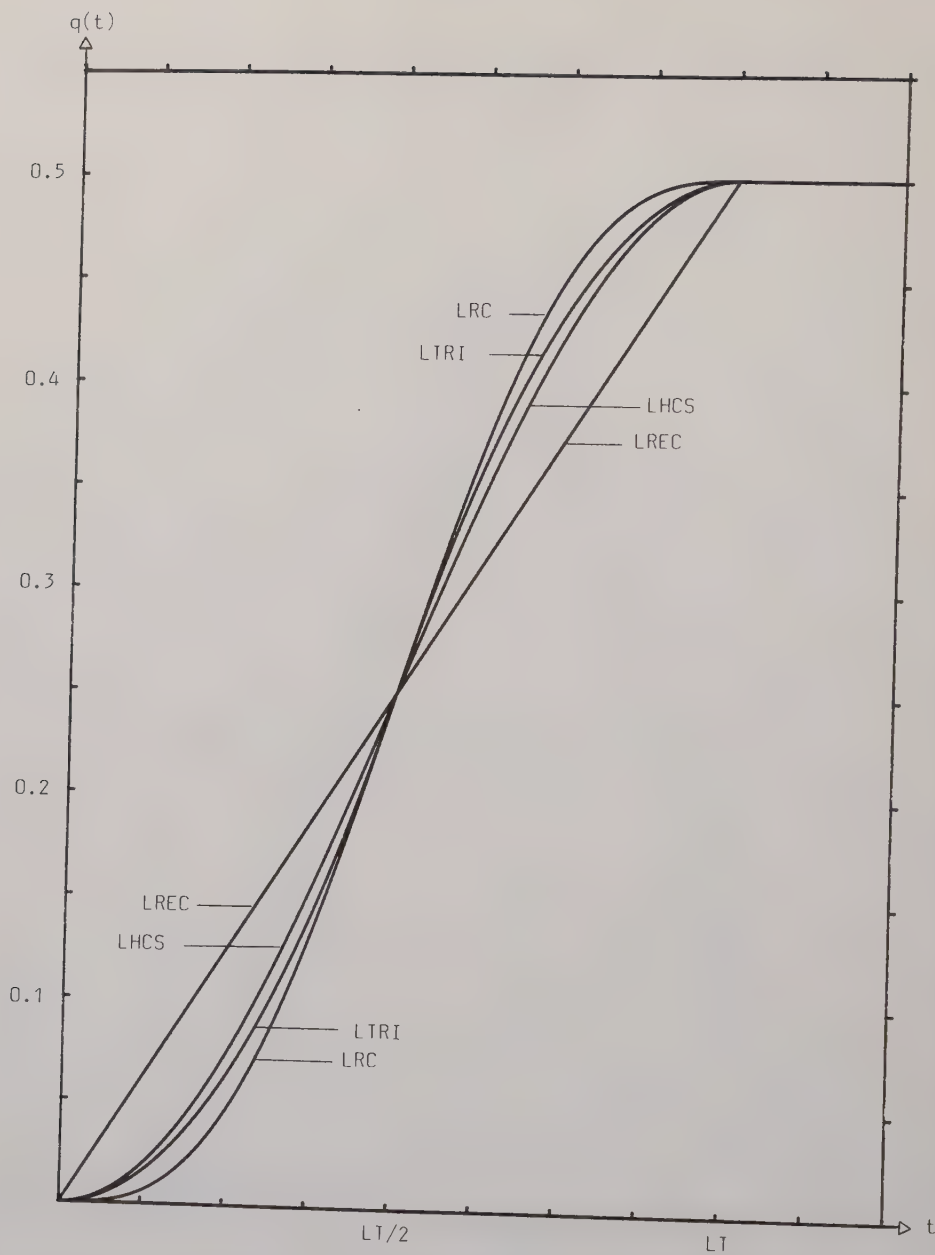


Figure 4. The LREC, LHCS, LTRI and LRC class of phase-responses $q(t)$.

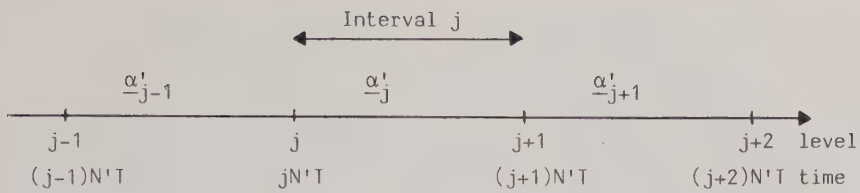


Figure 5. Illustrating the definition of level j and interval j .

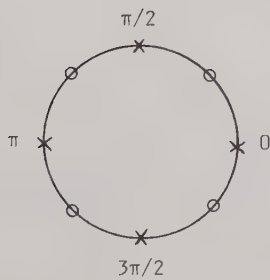


Figure 6. The two sets of phase states when $p=8$, see section II.

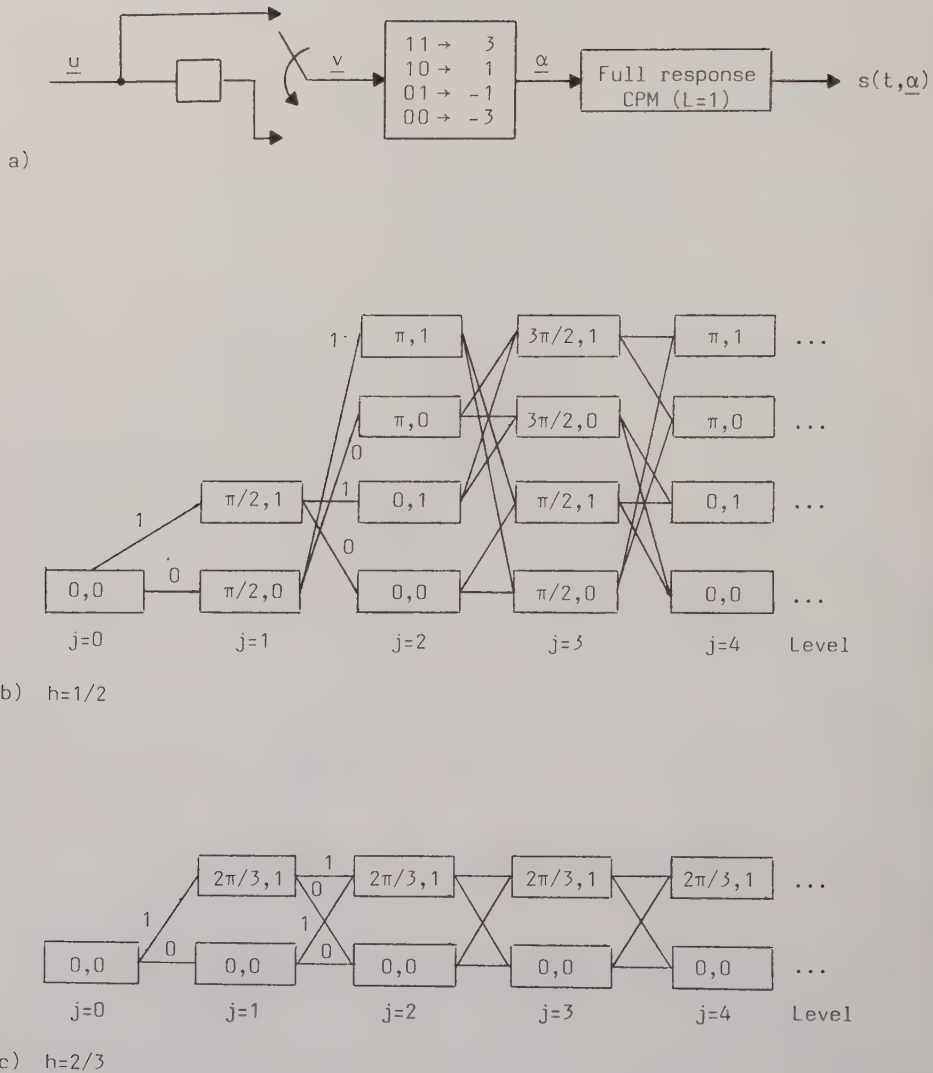
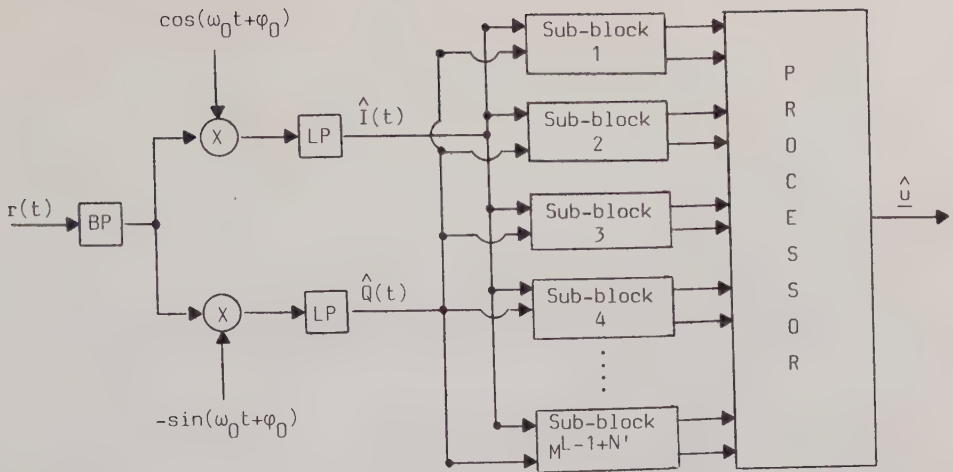
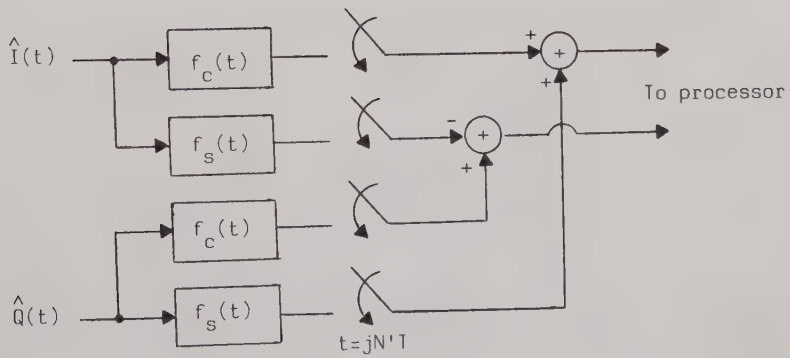


Figure 7. a) An example of a rate 1/2 convolutional encoder ($v=1$) combined with 4-level (nat.bin.) full response CPM.
 b) Signal trellis for the scheme in fig. 5a if $h=1/2$.
 c) Signal trellis for the scheme in fig. 5a if $h=2/3$.



a)



b)

Figure 8. a) A receiver structure.

b) The structure of the sub-blocks given in fig. 8a.
For details see Appendix B.

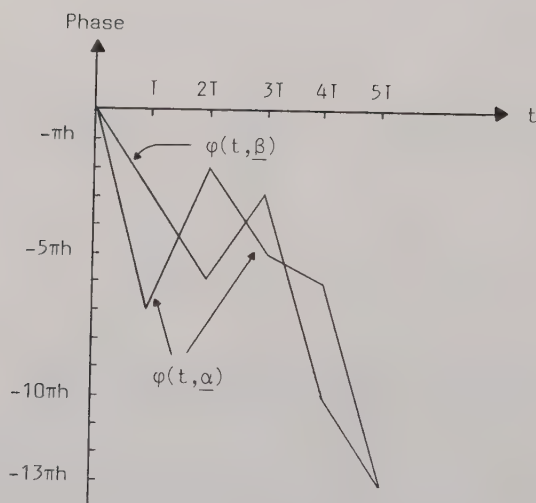
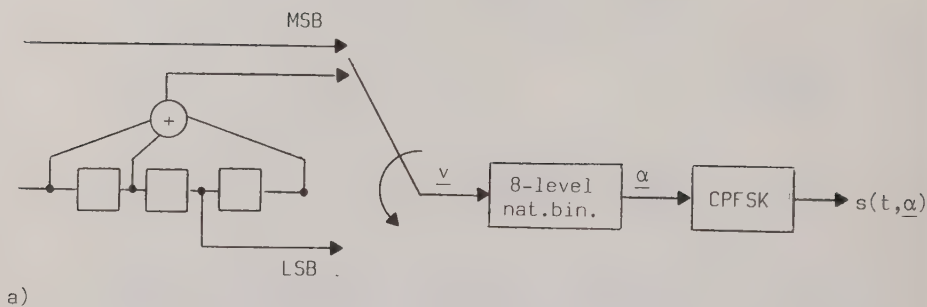


Figure 9. a) An example of a rate 2/3 convolutional encoder ($v=3$) combined with 8-level (nat.bin.) CPFSK modulation.
 b) Examples on phase functions for the scheme in fig. 9a. See section III for details.

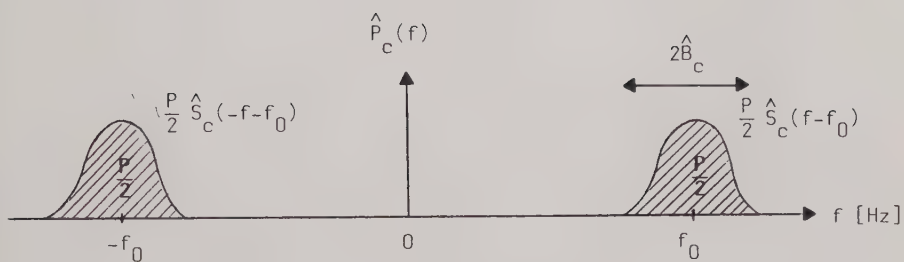


Figure 10. Illustrating the relationship between $\hat{P}_c(f)$ and $\hat{S}_c(f)$, see section IV.

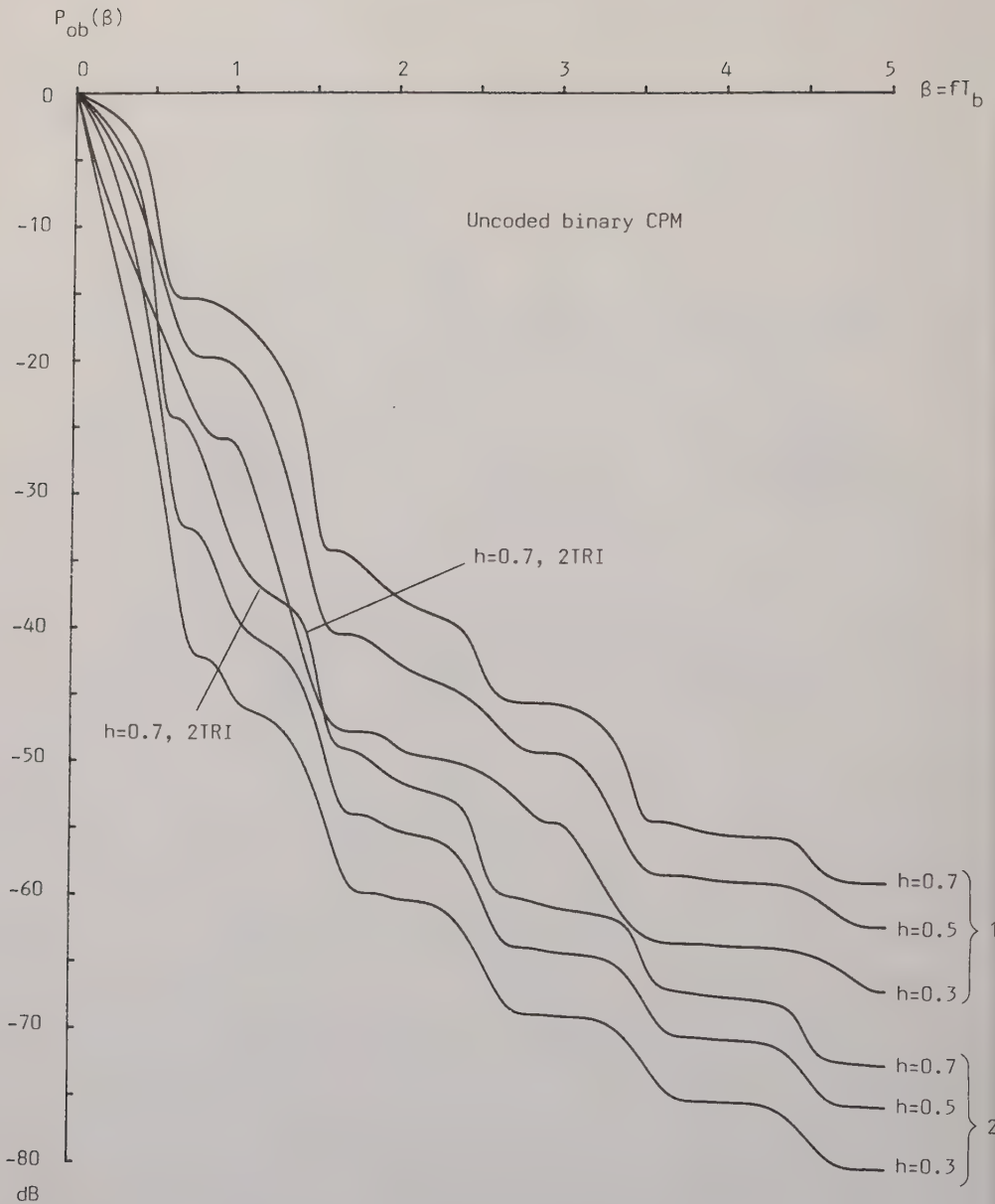


Figure 11. Fractional out of band power functions for 1TRI and 2TRI.
 $h=0.3, 0.5$ and 0.7 .

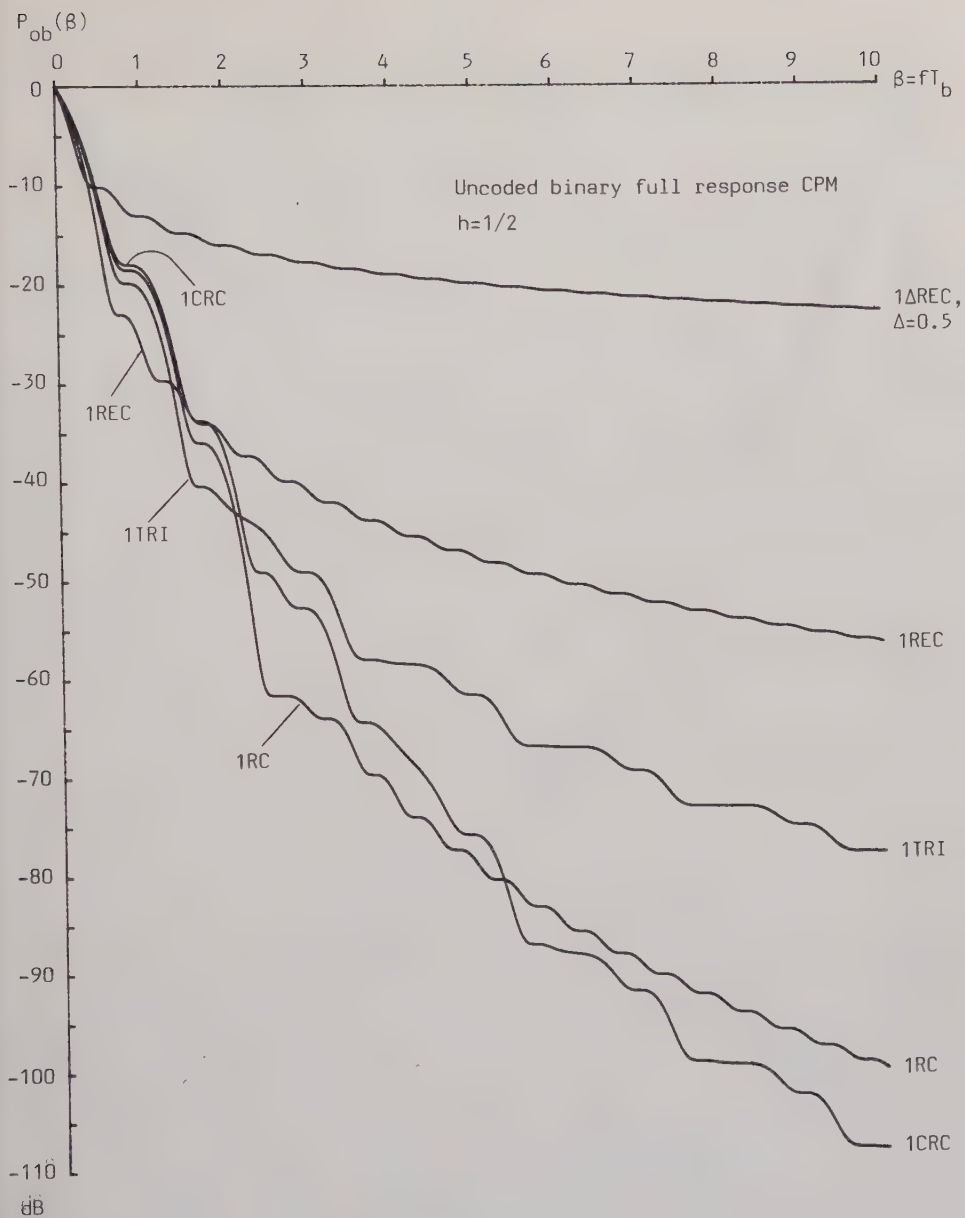


Figure 12: Fractional out of band power functions for different $g(t)$.
 $L=1$ and $h=1/2$.

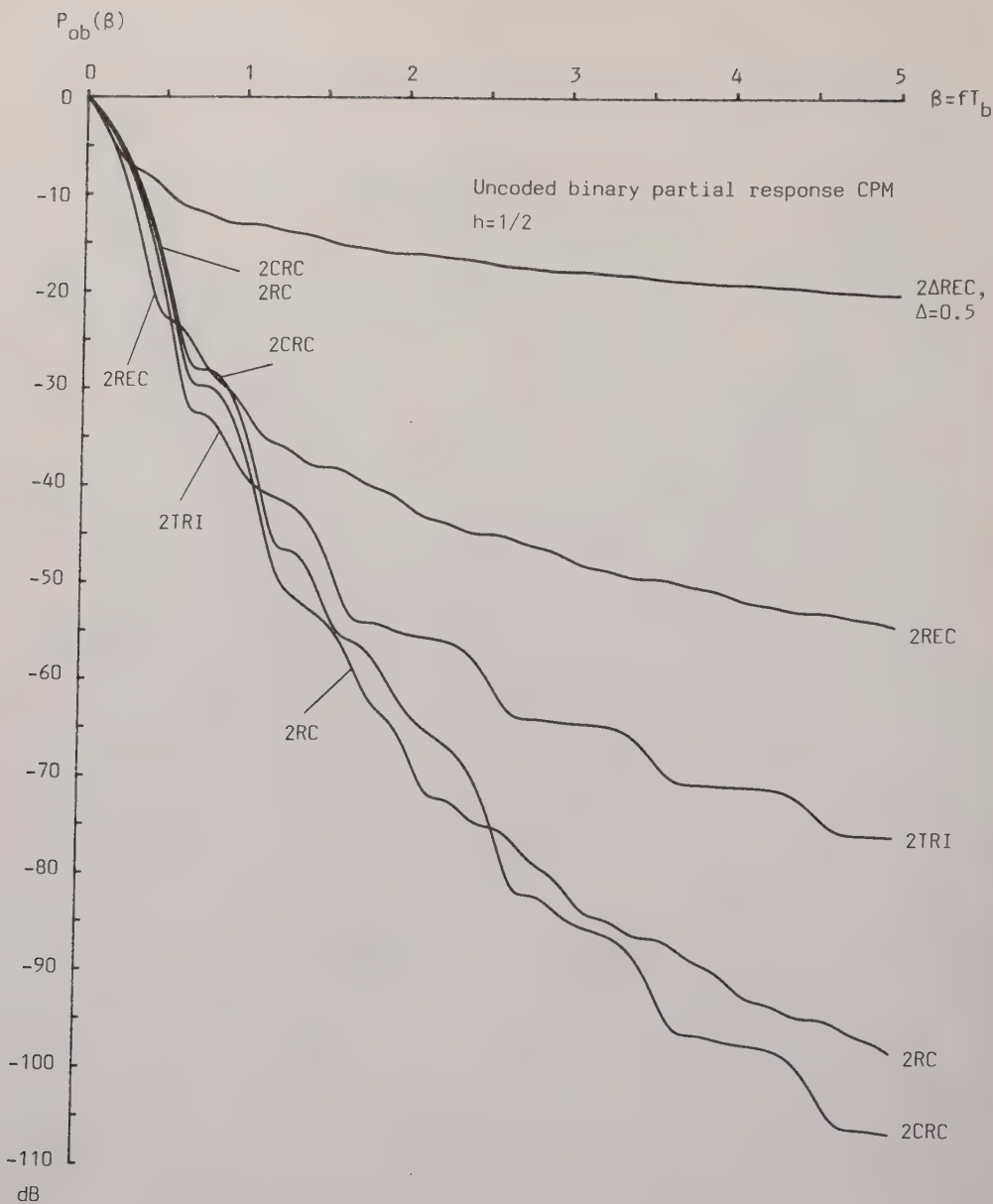


Figure 13. Fractional out of band power functions for different $g(t)$.
 $L=2$ and $h=1/2$.

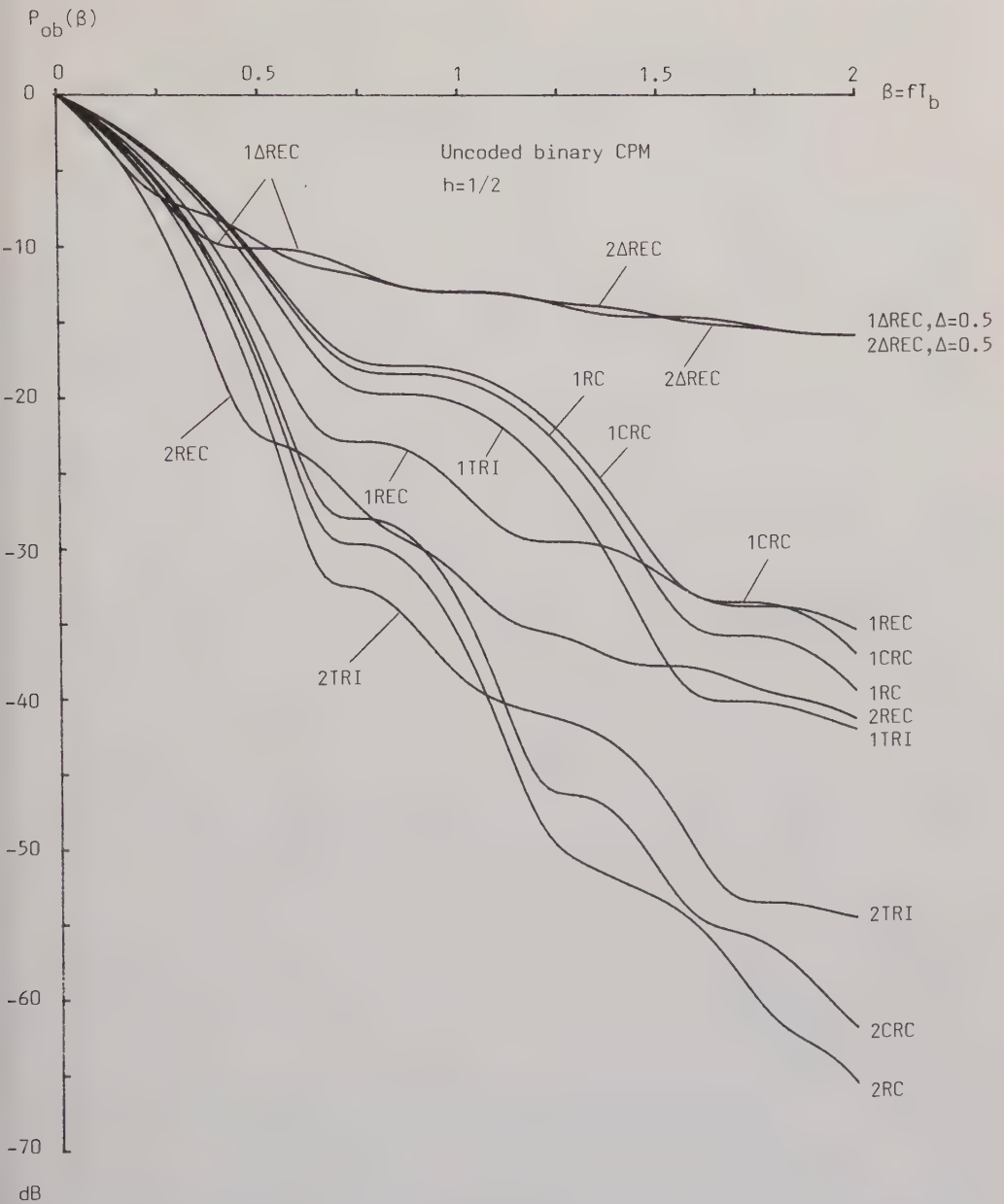


Figure 14. Fractional out of band power functions for different $g(t)$.
 $L=1, 2$ and $h=1/2$.

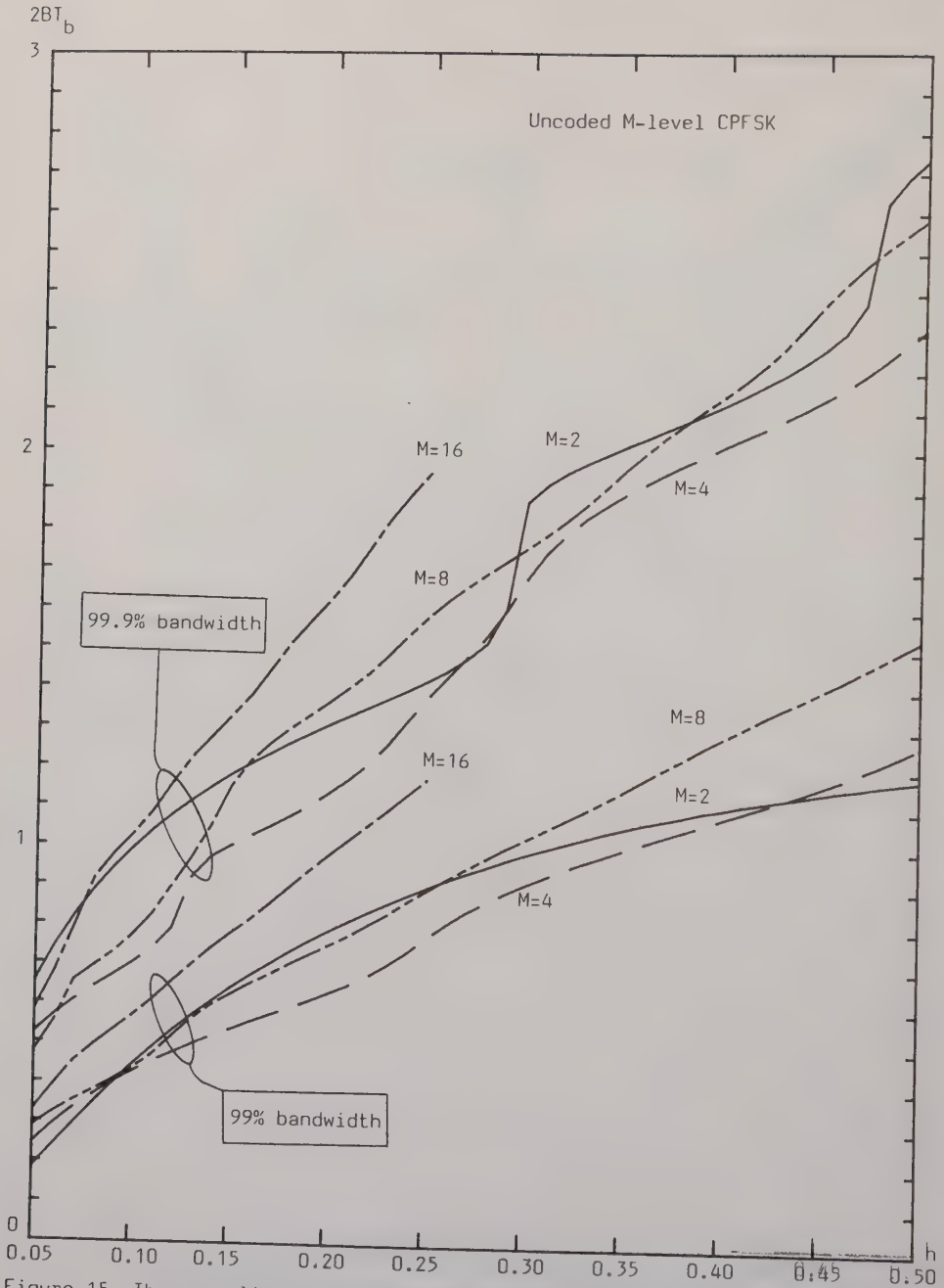


Figure 15. The normalized double-sided bandwidth $2BT_b$ as a function of the modulation index h when uncoded M-level CPFSK modulation is used: Two definitions of bandwidth are shown, the 99% in band power definition and the 99.9% in band power definition. See also table 2.

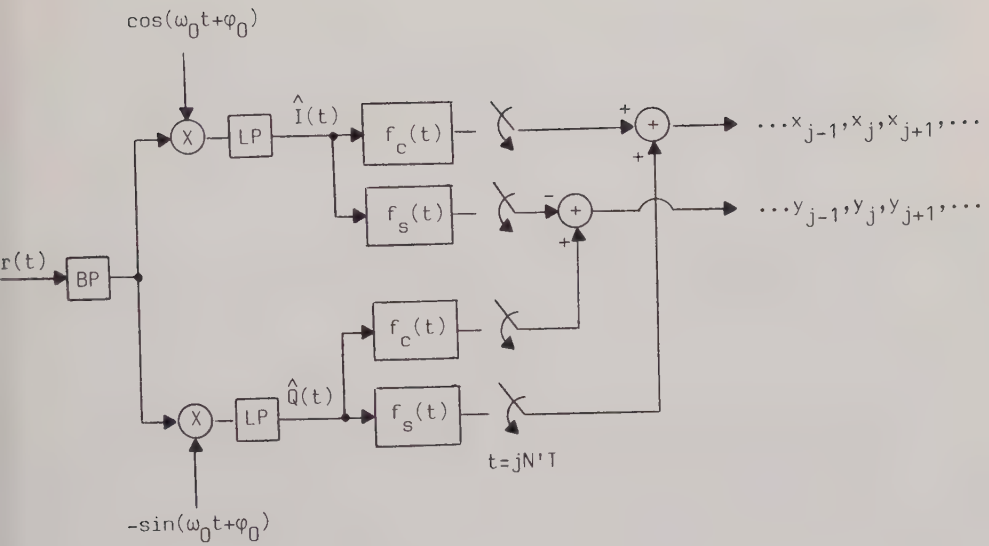


Figure B1. A possible structure to obtain the metrics, see Appendix B for details.

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ON CODED
CONTINUOUS PHASE MODULATION

PART II

MINIMUM EUCLIDEAN DISTANCE FOR COMBINATIONS
OF SHORT RATE $1/2$ CONVOLUTIONAL CODES AND CPFSK MODULATION

ABSTRACT

In this part we study the Euclidean distance properties of signals formed by a rate $1/2$ convolutional encoder followed by a binary or 4-level CPFSK modulator.

Especially, we have searched for those schemes that maximize the free Euclidean distance.

Symmetry properties are found, and these were used to reduce the number of convolutional encoders and mapping rules needed in the search.

The optimal codes are presented for the case of short rate $1/2$ codes combined with binary CPFSK modulation.

For short rate $1/2$ codes combined with 4-level CPFSK modulation, the optimal combination of code and mapper is presented.

Furthermore, the minimum Euclidean distance, as a function of the modulation index and the observation interval length is also studied in some detail.

It is found that the convolutional codes with optimum free Hamming distance do not in general give the optimum free Euclidean distance with CPFSK.

Schemes with a power gain in E_b/N_0 of 5-6dB, compared to the uncoded MSK scheme, have been found.

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I. INTRODUCTION

This part addresses the search for good schemes of combined convolutional codes and M-level CPM. Memory is introduced in the signal both by means of the convolutional code and the continuous phase in the CPM modulator. A Viterbi detector utilizes both types of memory. The advantages of CPM are its constant envelope modulation and low spectral tails [1]-[3]. By introducing convolutional coding, the error probability is improved.

In this part, it is assumed that the rate of the convolutional encoder equals $1/2$ and that binary or 4-level full response continuous phase frequency shift keying (CPFSK) is used. Thus, a bandwidth expansion by a factor of 2, compared to the corresponding uncoded CPFSK scheme, must be assessed in forming power bandwidth tradeoffs.

An additive white Gaussian noise (AWGN) channel is assumed. Furthermore, only coherent maximum likelihood sequence detection will be studied. The optimal combination of a noncatastrophic rate $1/2$ convolutional encoder and an M-level mapping rule means that their specific combination, for a given code memory v and a given modulation index h , gives the largest minimum Euclidean distance when combined with the CPFSK modulator. It is assumed that the observation interval over which the minimum distance is calculated is large enough to ensure that the maximum obtainable minimum distance (free Euclidean distance) is reached. We will also study in some detail minimum distance growth with the observation interval length. Examples of schemes that utilize similar principles are found in [4]-[5]. Independent work on coded CPM schemes can be found in [6]-[9].

Conventional convolutional codes are normally optimized in terms of their free Hamming distance, [10]-[11]. These codes can also be combined with CPM or, specifically, CPFSK. The important difference compared with the schemes in this part is that here the Euclidean distance for the combination of the channel coding and the modulation is optimized. By using the optimal codes in a Hamming distance sense, and even combining them with a soft decision demodulator, a nonoptimum combined coding and modulation scheme normally are the results.

Fig. 1 shows two examples of modulation schemes considered in this part. When a specific rate 1/2 convolutional encoder is referred to, an **octal** representation of the upper and lower polynomial is used. Fig. 1a shows the rate 1/2 (31,17)-encoder ($v=4$) combined with binary CPFSK modulation and fig. 1b shows the rate 1/2 (13,4)-encoder ($v=3$) combined with 4-level (nat.bin.) CPFSK modulation.

In this part it is assumed that $k=1$, $n=2$ and $M=2$ or 4. Furthermore, it is assumed that CPFSK modulation is used (1REC). The problem addressed here is to find those combinations of a noncatastrophic rate 1/2 convolutional encoder and mapping rule, such that, for fixed v and h , the free Euclidean distance $d_{\min}^2(h)$ is maximized. Formally, this can be written as

$$\max_{\left\{ \begin{array}{c} \text{mapping} \\ \text{rule} \end{array} \right\}} \max_{\left\{ \begin{array}{c} \text{noncatastrophic} \\ \text{rate 1/2 conv.} \\ \text{encoder} \end{array} \right\}} d_{\min}^2(h) \quad (1)$$

II. SYMMETRIES FOR THE COMBINATION OF CONVOLUTIONAL ENCODER AND MAPPING RULE

In this section, the combination of a convolutional encoder and a mapping rule is studied in more detail, because in the search for the best combinations (largest $d_{\min}^2(h)$) only as few combinations as necessary should be investigated. The total number of mapping rules is $M!$. By studying eq. (25) in Part I however, it is seen that the Euclidean distance would be the same if $\underline{\gamma}$ were replaced by $-\underline{\gamma}$. Consequently, only $M!/2$ mapping rules need to be investigated.

Rate 1/n Convolutional Encoder and Binary CPM:

In this binary case there is only one mapping rule of interest. This mapping rule is given in fig. 1a. The purpose of this section is to find symmetries in the combination of a rate 1/n convolutional encoder and a binary mapper.

Consider now two rate 1/n convolutional encoders, denoted by $\{g_{m,1}(i)\}$ and $\{\tilde{g}_{m,1}(i)\}$ respectively. If

$$\tilde{g}_{m,1}(i) = g_{n+1-m,1}(v-i) \quad ; \quad 1 \leq m \leq n, \quad 0 \leq i \leq v \quad (2)$$

then these two encoders, when combined with the binary mapper ($M=2$) and the same CPM modulator, yield the same free Euclidean distance, but, in general, not the same $d_{\min,N}^2$. The reason is that time reversal of phase and state mergers does not change $d_{\min}^2$, but, in general, changes $d_{\min,N}^2$. Note that this symmetry property is valid for any frequency pulse $g(t)$ and for any modulation index h in the CPM modulator. As an example of the above, fig. 2 shows the rate 1/2 (4,3)-encoder and the rate 1/2 (6,1)-encoder respectively. The symmetry property given above will now be proved.

Proof:

Those pair of data sequences, $\underline{u}$ and $\underline{w}$, giving a phase and state merger, can without loss of generality always be written as

$$\underline{u} = (\dots, 0, 0, 0, u_{-v}, u_{-v+1}, \dots, u_{-1}, u_0, u_1, \dots, u_{s-1}, 0, 0, 0, \dots) \quad (3)$$

and

$$\underline{w} = (\dots, 0, 0, 0, u_{-v}, u_{-v+1}, \dots, u_{-1}, w_0, w_1, \dots, w_{s-1}, 0, 0, 0, \dots) \quad (4)$$

respectively, where s is some integer. Now let the sequences $\underline{u}$ and $\underline{\tilde{u}}$,

$$\underline{\tilde{u}} = (\dots, 0, 0, 0, \tilde{u}_{-v}, \tilde{u}_{-v+1}, \dots, \tilde{u}_{-1}, \tilde{u}_0, \tilde{u}_1, \dots, \tilde{u}_{s-1}, 0, 0, 0, \dots) \quad (5)$$

be the input data sequences to the rate $1/n$ convolutional encoders $\{g_{m,1}(i)\}$ and $\{\tilde{g}_{m,1}(i)\}$ respectively. Denote the corresponding output sequences by $\underline{\xi}_m$ and $\underline{\tilde{\xi}}_m$, $m = 1, 2, \dots, n$. The i :th element in $\underline{\xi}_m$ and $\underline{\tilde{\xi}}_m$ is

$$\xi_{m,i} = \sum_{\ell=0}^v u_{i-\ell} g_{m,1}(\ell) \quad (6)$$

and

$$\tilde{\xi}_{m,i} = \sum_{\ell=0}^v \tilde{u}_{i-\ell} \tilde{g}_{m,1}(\ell) \quad (7)$$

respectively. It follows that if

$$\tilde{g}_{m,1}(i) = g_{n+1-m,1}(v-i) \quad ; \quad 1 \leq m \leq n, \quad 0 \leq i \leq v \quad (8)$$

and if

$$\tilde{u}_\ell = u_{r-\ell} \quad (9)$$

where $r = s-1-v$, then eq. (7) can be written as

$$\tilde{\xi}_{m,i} = \sum_{\ell=0}^v u_{r-i+\ell} g_{n+1-m,1}(v-\ell) = \sum_{\ell=0}^v u_{r-i+v-\ell} g_{n+1-m,1}(\ell) \quad (10)$$

By comparing eq. (6) and eq. (10) it is seen that the sequence $\tilde{\xi}_m$ is just a time-reversed variant of the sequence ξ_{n+1-m} ,

$$\tilde{\xi}_{m,i} = \xi_{n+1-m, s-1-i} \quad (11)$$

Therefore, after the switch, the sequence $\tilde{v}$ is a time-reversed variant of the sequence v . This is illustrated in fig. 3. Consequently, after the binary mapping rule, the channel symbol sequence $\tilde{\alpha}$ is only a time-reversed variant of the channel symbol sequence α .

Suppose now that the rate $1/n$ convolutional encoder $\{g_{m,1}(i)\}$ is used and that $\underline{u}$ and $\underline{w}$, eqs. (3)-(4), causes a phase and state merger. Then it is always possible to create a new rate $1/n$ convolutional encoder $\{\tilde{g}_{m,1}(i)\}$ according to eq. (8). When this encoder is fed with the sequences $\tilde{u}$ ($\tilde{u}_\ell = u_{r-\ell}$) and $\tilde{w}$ ($\tilde{w}_\ell = w_{r-\ell}$) respectively, a phase and state merger is produced which is just a time reversed variant of the phase and state merger produced by $\underline{u}$, $\underline{w}$ and $\{g_{m,1}(i)\}$. The Euclidean distance for a phase and state merger is not changed if it is time-reversed. Therefore, the rate $1/n$ convolutional encoders $\{g_{m,1}(i)\}$ and $\{\tilde{g}_{m,1}(i)\}$ must yield the same free Euclidean distance when combined with a binary mapper. Time-reversal of phase and state mergers affects in general $d_{\min,N}^2$, see the definition in Part I. This concludes the proof.

Rate $1/n$ convolutional encoder and 2^N -level CPM:

The purpose of this section is to find symmetries in the combination of a rate $1/n$ convolutional encoder and a 2^N -level mapper.

Consider now two rate $1/n$ convolutional encoders, denoted by $\{g_{m,1}(i)\}$ and $\{\tilde{g}_{m,1}(i)\}$ respectively. If

$$\tilde{g}_{m,1}(i) = g_{m,1}(v-i) \quad ; \quad 1 \leq m \leq n, \quad 0 \leq i \leq v \quad (12)$$

then these two encoders, when combined with the same 2^N -level mapper and the same CPM modulator, yield the same free Euclidean distance, but, in general, not the same $d_{\min,N}^2$. The reason is that time reversal of phase and state mergers does not change $d_{\min}^2$, but, in general, changes $d_{\min,N}^2$. Note that this symmetry property is valid for any frequency pulse $g(t)$ and for any modulation index h in the CPM modulator. As an example of the

above see fig. 4. This figure shows the rate $1/2$ (23,10)-encoder and the rate $1/2$ (31,2)-encoder respectively combined with 4-level CPM. The symmetry property given above will now be proved.

Proof:

The arguments and notations used in the previous proof will be used. Let the sequences $\underline{u}$ (eq. (3)) and $\tilde{\underline{u}}$ (eq. (9)) be the input data sequences to the rate $1/n$ convolutional encoders $\{g_{m,1}(i)\}$ and $\{\tilde{g}_{m,1}(i)\}$ respectively. It follows that if

$$\tilde{g}_{m,1}(i) = g_{m,1}(v-i); \quad 1 \leq m \leq n, \quad 0 \leq i \leq v \quad (13)$$

then the sequence $\tilde{\xi}_m$ is just a time-reversed variant of the sequence ξ_m ,

$$\tilde{\xi}_{m,i} = \xi_{m,s-1-i} \quad (14)$$

Therefore, after any 2^N -level mapping rule, the channel symbol sequence $\tilde{\underline{\alpha}}$ is only a time-reversed variant of the channel symbol sequence $\underline{\alpha}$.

Using the arguments in the previous proof, concludes this proof.

Rate 1/2 convolutional encoder combined with 4-level CPM:

The main purpose of this section is to reduce the number of mapping rules needed in the search, eq. (1). The 12 mapping rules of interest are given in table 1. Note that the natural binary mapper is denoted by Q1 in this table.

In this section it will be shown that any combination of a rate $1/2$ convolutional encoder and a 4-level mapping rule, can be transformed to another combination of a rate $1/2$ convolutional encoder and a 4-level mapping rule, where the mapping rule is either Q1 or Q2. Furthermore, the transformed combination will be identical to the original combination. Consequently, only mapping rules Q1 and Q2 are needed in the search, eq. (1). In this section it will also be shown that the combinations $\{\{g_{m,1}(i)\}, Q2\}$ and $\{\{g_{m,1}(i)\}, Q11\}$ are identical in terms of $d_{\min,N}^2$, if the rate $1/2$ convolutional encoder has an odd number of connections in its lower polynomial ($=g_{2,1}(i)$). This fact reduces the number of encoders needed in the search when Q2 is used, since the combination $\{\{g_{m,1}(i)\}, Q11\}$ can be transformed to an identical combination using mapping rule Q1.

Proof:

Consider in table 1 how the mapping rules Q1-Q15 and Q2-Q25 are created from each other by changing two levels in the previous mapping rule. This is indicated by dashed lines in table 1. Note that the two corresponding pairs of input coded symbols always have a "1" in the same position.

Take an arbitrary rate 1/2 convolutional encoder, say the (13,4)-encoder, and an arbitrary 4-level mapping rule, say Q11 defined in table 1. Let us now create a new rate 1/2 convolutional encoder from the (13,4)-encoder, which, when combined with the Q12 mapper, is identical to the combination {(13,4),Q11}. By studying the two mappers Q11 and Q12 respectively, it is seen that we can use the LSB to create a new encoder, see figs. 5a-5b. The combinations given in figs. 5a-5b are identical. Rewriting fig. 5b by adding the lower polynomial to the upper polynomial, yields the (17,4)-encoder, see fig. 5c. Thus, the combination {(17,4),Q12} is identical to the combination {(13,4),Q11}. Let us now create a new rate 1/2 convolutional encoder from the (17,4)-encoder, which, when combined with the Q13 mapper, is identical to the combination {(17,4),Q12}. By studying the two mappers Q12 and Q13 respectively, it is seen that we can use the MSB to create a new encoder, see figs. 5c-5d. The combinations given in figs. 5c-5d are identical. Rewriting fig. 5d, by adding the upper polynomial to the lower polynomial, yields the (17,13)-encoder, see fig. 5e. Thus, the combination {(17,13),Q13} is identical to the combination {(17,4),Q12}. Now it is clear that by successively adding the lower (upper) polynomial to the upper (lower) polynomial the following combinations are identical: {(13,4),Q11} $\Leftrightarrow$ {(17,4),Q12} $\Leftrightarrow$ {(17,13),Q13} $\Leftrightarrow$ {(4,13),Q14} $\Leftrightarrow$ {(4,17),Q15} $\Leftrightarrow$ {(13,17),Q1}. If the original mapping rule belongs to Q21-Q25, it is clear that the procedure above can be used to transform the original combination to an identical combination using mapping rule Q2. This concludes the proof.

Proof:

Now we will show that the combinations $\{g_{m,1}(i), Q2\}$ and $\{g_{m,1}(i), Q11\}$ are identical in terms of $d_{\min, N}^2$, if the rate 1/2 convolutional encoder has an odd number of connections in its lower polynomial ($=g_{2,1}(i)$).

Assume that the rate 1/2 convolutional encoder $\{g_{m,1}(i)\}$ has an odd number of connections in its lower polynomial ($=g_{2,1}(i)$). Let the sequence $\underline{u}$ be the input to the combination $\{g_{m,1}(i), Q2\}$ and denote by $\underline{\alpha}$ the corresponding channel symbol sequence. Now note that if the sequence $\underline{u} \oplus \underline{1}$ ($\underline{1}$ is the all-one sequence) is the input to the combination $\{g_{m,1}(i), Q11\}$, then the corresponding channel symbol sequence is $\underline{\alpha}$ or $-\underline{\alpha}$, depending on whether the number of connections in the upper polynomial ($=g_{1,1}(i)$) is even or odd respectively. Therefore, the combination $\{g_{m,1}(i), Q2\}$, with the input sequences $\underline{u}$ and $\underline{w}$, yields the same Euclidean distance as the combination $\{g_{m,1}(i), Q11\}$ with the input sequences $\underline{u} \oplus \underline{1}$ and $\underline{w} \oplus \underline{1}$ respectively. Consequently, $d_{\min, N}^2$ is the same for the two combinations. This concludes the proof.

III. AN ALGORITHM TO CALCULATE THE MINIMUM EUCLIDEAN DISTANCE

This section describes the algorithm used to calculate $d_{\min, N}^2(h)$.

Figs. 6a,b show a flowchart over this algorithm, which is a natural extension of the algorithm given in [2].

It is assumed that $d_{\min, N}^2(h)$ is to be calculated for the modulation indices $h = h_{\min}, h_{\min} + \Delta h, \dots, h_{\max} - \Delta h, h_{\max}$ and for $N = N', 2N', 3N', \dots$ until $d_{\min}^2(h)$ is reached. Note that $N'=1$ or $N'=2$ in this part.

The algorithm starts by upper bounding $d_{\min}^2(h)$ with a very large number, say $d_B^2(h) = 10000$. Then all the normalized squared Euclidean distances for $N=N'$ channel symbol intervals are calculated. Those pair of data symbol sequences having the normalized squared Euclidean distance less than or equal to the upper bound will be saved for further processing. The minimum of these saved distances is $d_{\min, N'}^2(h)$.

When $d_{\min, 2N'}^2(h)$ is to be calculated, only those pair of data symbol sequences saved when N was equal to N' are used. Each saved pair of data symbol sequences generates $2^{2k'}$ new pairs of data symbol sequences. The normalized squared Euclidean distance for all these new pairs of data symbol sequences must be calculated. Again, only those pair of sequences having the normalized squared Euclidean distance less than or equal to $d_B^2(h)$ are saved for further processing. The minimum of these saved distances is $d_{\min, 2N'}^2(h)$.

The algorithm continuous like this until $d_{\min}^2(h)$ has been found.

Note that the upper bound, $d_B^2(h)$, is tightened by the algorithm, since it can detect all necessary phase and state mergers. The algorithm stops when $d_{\min, N}^2(h) = d_B^2(h)$ which guarantees that $d_B^2(h) = d_{\min}^2(h)$. Thus, the free Euclidean distance has been found.

It is seen in eq. (25) in Part I, that the algorithm must save the information $\underline{\Omega}_j$ to be able to calculate the normalized squared Euclidean distance contribution over the j :th interval,

$$\underline{\Omega}_j = \left(\Delta\theta_j, x_{j-\eta}^u, u_{j-\eta}^u, \dots, u_{j-1}^u, x_{j-\eta}^w, w_{j-\eta}^w, \dots, w_{j-1}^w \right) \quad (15)$$

where the difference-phase state $\Delta\theta_j$ is

$$\Delta\theta_j = \left(2\pi h q (L T) \sum_{i=-\infty}^{jN'-L} \gamma_i \right) \bmod 2\pi \quad (16)$$

Compare eqs. (8)-(10) in Part I.

IV. NUMERICAL RESULTS

This section presents numerical results assuming a rate 1/2 convolutional encoder and binary or 4-level CPFSK modulation. Every combination of v and h requires a search (eq. (1)), and each search produces the maximum $d_{\min}^2(h)$, together with those combinations of a convolutional encoder and a mapping rule, which maximize $d_{\min}^2(h)$. These combinations will be referred to as optimal combinations. The results are shown in tables 3-4 and tables 7-9. The N -value given in these tables is the value of N for which $d_{\min}^2(h) = d_{\min,N}^2(h)$. It should be observed that these tables are written in a minimal way. If, for example, all optimal combinations using mapping rules Q1 or Q2 are wanted when $v=1$ and $h=0.15$, table 7 and the results in section II must be used. Doing this, it is seen that the combinations $\{(1,2),Q1,4\}$ and $\{(2,1),Q1,?\}$ are the only optimal combinations. No calculation has been performed for the latter combination. Doing the same example when $v=2$ and $h=0.35$, yields the optimal combinations $\{(3,4),Q1,40\}$, $\{(3,7),Q2,40\}$, $\{(6,1),Q1,?\}$, and $\{(6,7),Q2,?\}$.

There is quite a lot of figures associated with this section. Most of them show $d_{\min,N}^2(h)$ as a function of h , with N as a parameter. These figures do also illustrate that the minimum Euclidean distance depends, in a non-trivial way, on the convolutional encoder.

Minimum distance pairs (phase and state mergers with minimum Euclidean distance) are also identified for a few selected schemes. The properties of these pairs will be studied in more detail in Part IV, where the bit error probability is considered.

IV.1 Rate 1/2 Convolutional Encoder and Binary CPFSK Modulation

In this case the search is only over the noncatastrophic convolutional encoders since there is only one mapping rule of interest. The symmetry property discussed in section II was used to reduce the search. The modulation indices $h = 0.05, 0.10, \dots, 0.95$ and $h=1$ are studied, and $1 \leq v \leq 4$. The optimum $d_{\min}^2(h)$ for different v and h are given in tables 3-4, together with the corresponding optimal convolutional encoders. The second argument within the brackets in these tables is the minimum even N for which $d_{\min,N}^2(h) = d_{\min}^2(h)$. See also fig. 47. As a reference, $d_{\min}^2(h)$ for the uncoded binary CPFSK scheme is given in table 2.

Note that the free Hamming distance for the convolutional code equals $2d_{\min}^2(1)$. This is so because for $h=1$ there is no memory introduced by the CPFSK modulator. A phase merger takes place after each channel symbol interval. Consequently, when $h=1$, the optimal convolutional encoders from the free Euclidean distance point of view are identical to the optimal convolutional encoders from the free Hamming distance point of view.

$v=1$:

Figs. 7-12 show $d_{\min,N}^2(h)$ for all 6 $v=1$ encoders. See for example fig. 10 which shows $d_{\min,N}^2(h)$ when the (2,1)-encoder is used. It is seen that for this encoder and with modulation indices less than ≈ 0.35 , the minimum distance reaches its maximum value, $d_{\min}^2(h)$, when $N=8$, but for modulation indices around 0.5, 0.75 and 1, the growth with N is very slow. This is a typical behaviour around h values for which a phase and state merger has occurred modulo 2π . It is also seen that with this encoder and modulation index $1/2$ the minimum normalized squared Euclidean distance, $d_{\min}^2(1/2)$, is equal to 3, and that the minimum even N for which this is reached is 6. Furthermore, for the code in fig. 10, $d_{\min}^2(1) = 1$. Consequently the free Hamming distance of this code equals 2. In general, the minimum Hamming distance (and the free Hamming distance with sufficient N) is obtained by looking at $h=1$ in the distance graphs. The normalized squared Euclidean distance value at $h=1$ must be multiplied by 2 due to the distance difference between PSK and CPFSK ($h=1$).

By studying figs. 7-12 it is found that the optimal $v=1$ encoders, when the modulation index equals $1/2$, are the (2,3) and (3,1)-encoders (figs. 11-12 respectively) which have $d_{\min}^2(1/2) = 4$. The corresponding N -values are 12 and 10, making the (3,1)-encoder the preferable one. To find the optimal encoders for other modulation indices fig. 13 can be useful. This figure shows $d_{\min,40}^2(h)$ for all 6 $v=1$ encoders. From figs. 7-13 the optimal encoders for h -values in the interval $0 \leq h \leq 1$ can be found, and these are listed together with the corresponding optimum $d_{\min}^2(h)$, for some specific h -values in table 3. See also fig. 47. Note that the optimal encoders depend on the modulation index. For example, when the modulation index is less than ≈ 0.31 , the (1,2)-encoder is the best, but when the modulation index is greater than ≈ 0.31 the (2,1)-encoder is the best except at $h=1/2$ and $h=1$ where the (2,3), (3,1) and (2,3), (3,1), (1,3), (3,2)-encoders are the best respectively. In addition, when $h=0.75$ the (2,3) and (3,1)-encoders are just as good as the (2,1)-encoder.

v=2:

Figs. 14-37 show $d_{\min,N}^2(h)$ for all 24 v=2 encoders. See for example figs. 28-29 which show $d_{\min,N}^2(h)$ for the (4,3) and (6,1)-encoder respectively. These two encoders have the same $d_{\min}^2(h)$, see section II, but not the same $d_{\min,N}^2(h)$. It is seen in figs. 28-29 that the minimum distance growth with N is different for these two encoders. As an example, when $h=1/2$ both encoders have $d_{\min}^2(1/2) = 6$, but the (4,3)-encoder reaches this value when $N=20$ and the (6,1)-encoder when $N=22$, making the (4,3)-encoder the preferable one for this h-value. Furthermore, $d_{\min}^2(1) = 1.5$ and consequently the minimum Hamming distance is equal to 3 for these two encoders (while the best free Hamming distance is 5 for v=2, [10]-[11]). Figs. 34-35 show $d_{\min,N}^2(h)$ for the (5,7)-encoder and for the (7,5)-encoder respectively. These encoders have the best free Hamming distance ($d_f=5$) for v=2. See $h=1$ in figs. 34-35. It is clear from these figures that the (5,7) and the (7,5)-encoder are not the best encoders, in terms of the free Euclidean distance, in the neighbourhood of $h \approx 1/2$ for example. The (5,7) and the (7,5)-encoder only achieves $d_{\min}^2(1/2) = 4$ while the overall peak for v=2 is $d_{\min}^2(1/2) = 6$, see fig. 38.

Fig. 38 shows $d_{\min,40}^2(h)$ for all v=2 encoders, and together with figs. 14-37 it is possible to find the optimal v=2 encoders for modulation indices in the interval $0 \leq h \leq 1$. Some of these are listed in table 3. See also fig. 47. It is found that the considered (4,3) and (6,1)-encoders are the optimal encoders for modulation indices up to ≈ 0.76 with a possible exception at $h=2/3$. When the modulation index equals 0.80, the (4,5) and (5,1)-encoders are optimal, and when $h=0.90, 0.95$ or 1, the optimal encoders are the (5,7) and (7,5)-encoders. Note that, as is seen in table 3, the best v=1 encoder is better than the best v=2 encoders when the modulation index is very small, but when h is increased the best v=2 encoders are better than the best v=1 encoders.

v=3:

In this case the search for the optimal encoders has been restricted to the specific modulation indices $h = 0.05, 0.10, 0.15, \dots, 0.95$ and 1. The results are shown in table 4, where the optimum $d_{\min}^2(h)$ and the corresponding optimal encoders are given respectively. See also fig. 47. The (12,7) and (10,3)-encoders are examples of the best encoders for modulation indices in the intervals $0 \leq h \leq 1/2$ and $1/2 \leq h \leq 0.70$ respectively. The minimum distance growth with N has been plotted for these encoders in figs. 39-40 respectively. The optimal encoder from the free Hamming distance point of view is the (17,15)-encoder, [10]-[11], and this encoder belongs naturally to the optimal encoders when $h=1$. The minimum distance growth with N for this encoder is shown in fig. 41.

It should be noted that for modulation indices less or equal $h=1/2$ there is very little, or no, improvement in the optimum $d_{\min}^2(h)$, compared to the $v=2$ case. Furthermore, for very small modulation indices the best $v=1$ encoder is better than both the best $v=2$ and $v=3$ encoders.

v=4:

The results are shown in table 4, where the optimum $d_{\min}^2(h)$ and where the corresponding optimal encoders are given. It is seen in table 4 and in fig. 47 that there is a significant increase in the optimum normalized squared free Euclidean distance when compared with schemes with smaller v values. Specifically it is seen that $d_{\min}^2(1/2)$ equals 8. On the other hand, the value of N for which $d_{\min,N}^2(h) = d_{\min}^2(h)$ is also increased. The minimum distance growth with N , $d_{\min,N}^2(h)$, has been tabulated for some selected modulation indices and $v=4$ encoders in tables 5-6.

Summary of numerical results:

The optimum normalized squared free Euclidean distance for different v and h are plotted in figure 47 and tabulated in tables 3-4. The corresponding optimal encoders are also given in these tables. As a reference the uncoded CPFSK scheme is also given (table 2). It is seen in figure 47 that the minimum distance is increased when v is increased, except for very small modulation indices. It is also seen in this figure that for modulation indices less or equal $h=1/2$ there is no use to increase v from 2 to 3. The largest value found is $d_{\min}^2(1/2) = 8$ which is reached when $v=4$ and $h=1/2$. It is also found that the optimal encoder depends on h and that the value of N for which $d_{\min,N}^2(h) = d_{\min}^2(h)$ is increased when v is increased.

Figs. 42-46 show examples of minimum distance pairs (phase and state mergers with minimum Euclidean distance) for some of the optimal schemes found above.

When $v=1$ and $h \leq 0.30$, the (1,2)-encoder is optimal, see fig. 42. This figure shows a minimum distance pair of length $6T$. It is also found that for the scheme in fig. 42, with the modulation index $h=0.20$, all minimum distance pairs have length $6T$.

The optimal $v=1$ encoder when $h=1/2$ is the (2,3)-encoder, see fig. 43. This figure shows two minimum distance pairs. Note that these two pairs have different lengths, $6T$ and $8T$ respectively, and that the normalized squared Euclidean distance is the same when $h=1/2$ ($=4$). It is also found that for the scheme in fig. 43, with the modulation index $h=1/2$, minimum distance pairs do only exist with lengths $6T$ and $8T$.

When $v=2$ and $h \leq 3/4$ the (4,3)-encoder is optimal, see figs. 44-45. Fig. 44 shows two minimum distance pairs if $h \leq 1/2$. It is seen in this figure that the pairs are of length $14T$ and $16T$, and that there are time-segments of length $4T$ and $6T$ respectively where the Euclidean distance contribution is zero. Furthermore, for the scheme in fig. 44 with $h=0.20$, minimum distance pairs do only exist with lengths $14T$ and $16T$. Fig. 45 shows three additional minimum distance pairs if $h=1/2$. Note that these three pairs have the same Euclidean distance as the two pairs in fig. 44 if $h=1/2$, and note also that the reason for this is modulo- 2π effects. It is also found that for the scheme in fig. 45, with $h=1/2$, minimum distance pairs do only exist with lengths $8T$, $10T$, $12T$, $14T$ and $16T$.

The optimal $v=4$ encoder when $h=1/2$ is the (31,17)-encoder, see fig. 46. This figure shows two minimum distance pairs. It is seen in this figure that the two pairs have length $12T$ and $16T$ respectively. Furthermore, it is found that for the scheme in fig. 46, with $h=1/2$, minimum distance pairs do only exist with lengths $12T$, $14T$, $16T$, $20T$ and $24T$.

Note that a phase and state merger found with CPFSK modulation, also is very useful when the CPFSK frequency pulse is replaced with another frequency pulse $g(t)$, see section III in Part I.

IV.2 Rate 1/2 Convolutional Encoder and 4-level CPFSK

In this case the search (eq. (1)) is over all noncatastrophic convolutional encoders and over all 4-level mapping rules. Symmetry properties discussed in section II were used to reduce the search. The modulation indices $h = 0.05, 0.10, \dots, 0.55$ and $h=0.60$ are studied and $1 \leq v \leq 4$. The optimum $d_{\min}^2(h)$ for different v and h are given in tables 7-9, together with the corresponding optimal combinations of a rate 1/2 convolutional encoder and a 4-level mapping rule. The third argument within the brackets in these tables is the minimum N for which $d_{\min,N}^2(h) = d_{\min}^2(h)$.

As a reference, the minimum normalized squared Euclidean distance growth with N , $d_{\min,N}^2(h)$, for the uncoded 4-level CPFSK scheme is shown in fig. 48, see also table 2.

$v=1$:

It is seen from table 7 and from fig. 66 that the free Euclidean distance is significantly increased, compared with the uncoded case, when the modulation index $h \leq 0.35$. Furthermore, when $h=1/2$, $d_{\min}^2(1/2)$ is increased from 2 to 3. Note that when $h \leq 0.45$, no combination using mapping rule Q2 is optimal, and when $h = 0.55$ and 0.60 no combination using mapping rule Q1 is optimal. In figs. 49-53 $d_{\min,N}^2(h)$ as a function of h , with N as parameter, is plotted for the combinations $\{(1,2),Q1\}$, $\{(2,1),Q1\}$, $\{(1,3),Q2\}$, $\{(2,3),Q2\}$ and $\{(3,1),Q1\}$ respectively. Comparing fig. 49 with fig. 50 it is seen that, for small modulation indices, the $\{(2,1),Q1\}$ combination is the preferable one, since it reaches $d_{\min}^2(h)$ faster (smaller N). It is also seen in fig. 49 that around the modulation indices $h = 1/3$, $2/3$ and $h=1$, $d_{\min,N}^2(h)$ grows very slow with N . This is a normal behaviour around so-called weak modulation indices. A different ("more stable") structure in the minimum Euclidean distance growth with N is shown in figs. 51-52. To compare the uncoded 4-level CPFSK scheme, fig. 48, with the coded schemes in figs. 49-53, fig. 54 is studied. This figure shows the tight (almost everywhere) upper bounds (envelopes), $d_B^2(h)$, in figs. 48-53. It is seen in fig. 54 that when the modulation index is small, or equals $1/2$, an increase in the free Euclidean distance can be obtained, see also fig. 66.

$v=2$:

Increasing the number of delay elements in the convolutional encoder from one to two, does also increase the optimum free Euclidean distance, as is seen in table 7 and in fig. 66. For modulation indices $h \leq 0.30$ the $\{(7,2),Q1\}$ and the $\{(7,5),Q2\}$ combinations are optimal, and for modulation indices around $h=1/2$ the $\{(1,7),Q1\}$ and $\{(1,6),Q2\}$ combinations are optimal. When $h=1/2$ the optimum $d_{\min}^2(1/2) = 5$, and this is the largest value on $d_{\min}^2(h)$ found when $v=2$.

Note that whenever a combination $\{g_{m,1}(i),Q11\}$ is optimal, the combination $\{g_{m,1}(i),Q2\}$ is also optimal, even when the number of connections in the lower polynomial is even (compare with section II). Furthermore, the two combinations are almost (but not exactly) identical from the $d_{\min,N}^2(h)$ point of view. Examples of this will be given below. From the $v=1$ case however, it is known that this is not a general property. Fig. 49 and fig. 51, for example, show that the two combinations $\{(1,2),Q1\}$ and $\{(1,3),Q2\}$ have different free Euclidean distance.

In figs. 55-60 $d_{\min,N}^2(h)$ is plotted for the combinations $\{(7,2),Q1\}$, $\{(7,5),Q2\}$, $\{(1,7),Q1\}$, $\{(1,6),Q2\}$, $\{(2,7),Q1\}$ and $\{(2,5),Q2\}$ respectively. Figs. 55-56 show that the two combinations $\{(7,2),Q1\}$ and $\{(7,5),Q2\}$ have almost the same $d_{\min,N}^2(h)$, and that they seem to have the same free Euclidean distance. By studying these two figures carefully, for example around $h=0.4$, it is seen that they have different minimum distance growth $d_{\min,N}^2(h)$. Figs. 57-58 show that the two combinations $\{(1,7),Q1\}$ and $\{(1,6),Q2\}$ seem to have identical minimum distance growth $d_{\min,N}^2(h)$. Studying figs. 59-60 around $h=0.4$, for example, it is found that the two combinations $\{(2,7),Q1\}$ and $\{(2,5),Q2\}$ have different minimum distance growth $d_{\min,N}^2(h)$. However, these two combinations seem to have identical free Euclidean distance.

To compare the uncoded 4-level CPFSK scheme, fig. 48, with the coded schemes in figs. 55-60, fig. 61 is studied. This figure shows the tight (almost everywhere) upper bounds (envelopes), $d_B^2(h)$, in figs. 55-60 and in fig. 48. It is seen in fig. 61 that a significant increase in the free Euclidean distance can be obtained for almost all modulation indices in the interval $0 \leq h \leq 1$. See also fig. 66. On the other hand, the corresponding N -values are also, normally, significantly larger for the coded schemes.

v=3:

Using 3 delay elements in the convolutional encoder increases the optimum free Euclidean distance significantly compared to the $v=2$ case, as is seen in table 8 and in fig. 66. The number of optimal combinations are also increased. Table 8 shows that when the modulation index $h \leq 1/4$ the combinations $\{(13,4),Q1\}$ and $\{(13,17),Q2\}$ are optimal. Increasing the modulation index to the interval $0.30 \leq h \leq 0.40$ yields the combination $\{(3,10),Q1\}$ to be optimal, and when h is around $h=1/2$ the $\{(1,15),Q1\}$ and $\{(1,14),Q2\}$ combinations are optimal. When the modulation index equals $1/2$ the optimum $d_{\min}^2(1/2) = 6$ and this is the largest value found on $d_{\min}^2(h)$ when $v=3$. Furthermore, for this h ($1/2$) there is quite a lot of optimal combinations.

Table 10 shows $d_{\min,N}^2(h)$ for some selected modulation indices, when the combination $\{(13,4),Q1\}$ or $\{(13,17),Q2\}$ is used respectively.

v=4:

For $v=4$, a complete search has only been done for $h=1/4$. The optimum $d_{\min}^2(1/4) = 6.151$ and the corresponding optimal combinations are the $\{(23,10),Q1\}$ combination and the $\{(23,33),Q2\}$ combination. See table 9 and fig. 66. For many of the other modulation indices, combinations have been found which significantly increase the free Euclidean distance compared with the $v=3$ case. As an example, when $h=1/2$, the $\{(4,23),Q1\}$ combination achieves $d_{\min}^2(1/2) = 7$.

Summary of numerical results:

The optimum $d_{\min}^2(h)$ for different v and h are plotted in fig. 66 and is tabulated in tables 7-9. The corresponding optimal combinations of a rate $1/2$ convolutional encoder and 4-level mapping rule are also given in these tables. It is seen in fig. 66 that the free Euclidean distance is increased when v is increased. The largest value found is $d_{\min}^2(1/2) = 7$ which is reached when $v=4$ and when $h=1/2$.

Figs. 62-65 show examples of minimum distance pairs for some of the optimal schemes found above.

When $v=1$ and $h \leq 1/2$ the $\{(1,2),Q1\}$ combination is optimal, see fig. 62.

This figure shows a minimum distance pair, when $h=1/4$, of length $3T$. It is also found that for the scheme in fig. 62, with the modulation index $h=1/4$, all minimum distance pairs have length $3T$.

The optimal combination when $v=2$ and $h \leq 0.30$ is the $\{(7,2),Q1\}$ combination, see fig. 63. This figure shows a minimum distance pair when $h=1/10$ and when $h=1/4$ respectively. The minimum distance pair for $h=1/10$ given in fig. 63 is actually also a minimum distance pair for $h=1/4$. Furthermore, for the scheme in fig. 63, with the modulation index $h=1/10$ or $h=1/4$, all minimum distance pairs have length $6T$.

When $v=3$ and $h \leq 1/4$ the $\{(13,4),Q1\}$ combination is optimal, see fig. 64. This figure shows a minimum distance pair, when $h=1/4$, of length $8T$. It is also found that for the scheme in fig. 64, with the modulation index $h=1/4$, all minimum distance pairs have length $8T$.

The optimal combination when $v=4$ and $h=1/4$ is the $\{(23,10),Q1\}$ combination, see fig. 65. This figure shows a minimum distance pair, when $h=1/4$, of length $6T$. This pair has a much larger Euclidean distance than the minimum distance pairs, for $h=1/4$, given in figs. 62-64. Furthermore, for the scheme in fig. 65, with the modulation index $h=1/4$, all minimum distance pairs have length $6T$.

V. DISCUSSION AND CONCLUSIONS

It is well-known in the literature [10]-[11], [13] that the error probability for large SNR (E_b/N_0) is dominated by the free Euclidean distance $d_{\min}^2$, i.e., by the term $Q(\sqrt{d_{\min}^2 E_b/N_0})$. For comparison, the behavior of a rate 1/2 convolutional code combined with a BPSK modem is $Q(\sqrt{d_f E_b/N_0})$, where d_f is the free Hamming distance. From the results above and from the literature, we obtain table 11. In general the optimal code for d_f is different from the optimal code for $d_{\min}^2$. It is expected that the optimum $d_{\min}^2(h)$ will grow with increasing v . Symmetry properties were used to reduce the search. The computational problems grow fast, however, and for the moment we have not reached further than table 11.

Minimum distance pairs are identified for a few selected schemes. The properties of these pairs will be studied in more detail in Part IV, where the bit error probability is considered.

The results above show that it is possible to achieve considerable distance gains with coding. However, bandwidth is also increased, compared to the uncoded scheme. Fig. 67 shows a comparison in terms of both power in decibels relative to MSK, i.e., $10\log_{10}(d_{\min}^2/2)$, and bandwidth (estimated). The bandwidth is defined as the normalized double-sided bandwidth for which 99 percent of the total power lies within the band (section IV in Part I). Thus, all schemes in fig. 67 have the same data rate. They are also all compared as to E_b/N_0 at high SNR's. It should be pointed out that no spectral calculations have actually been carried out for the coded CPFSK schemes above. We have estimated spectra by using the uncoded CPFSK spectra at a given h and expanding them by means of $1/R_c$ (section IV in Part I).

The coded CPFSK schemes are compared with uncoded binary and 4-level CPFSK schemes, and it is quite evident that the coded CPFSK schemes are increasingly more power efficient with increasing code memory v (increasing complexity). It is also seen in fig. 67 that schemes have been found which are much better than the MSK-scheme in terms of E_b/N_0 and have approximately the same spectral efficiency. Compare, for example, the $v=4$, $h=0.2$ coded 4-level CPFSK scheme with MSK.

To achieve still more spectrally efficient schemes, it is necessary to study a different family of coded CPM schemes. An example on this is high rate convolutional encoders and multilevel CPFSK schemes.

Recently, several approaches to good combined channel coding and digital modulation schemes have been presented. Examples of schemes studied in [4] are rate $2/3$ and rate $3/4$ convolutional codes combined with 8-PSK and 16-QASK respectively. Rate $3/4$ codes and 16-PSK are considered in [14]. Independent work on coded CPM schemes are found in [6]-[7] where also coded partial response CPM is studied. See also [8]-[9].

Input to the mapper		Output channel symbol											
MSB	LSB	Q1	Q11	Q12	Q13	Q14	Q15	Q2	Q21	Q22	Q23	Q24	Q25
1	1	3	-1	-1	-3	-1	-1	3	1	-3	3	1	-3
1	0	1	-3	3	-1	-1	-1	1	3	3	-3	-3	1
0	1	-1	-1	1	1	3	3	-3	-3	1	1	3	3
0	0	-3	-3	-3	-3	-3	-3	-1	-1	-1	-1	-1	-1

Table 1. 4-level mapping rules. The dashed lines indicate differences between mapping rules.

h	$d_{\min}^2$ for uncoded M-level CPFSK modulation	
	Binary	4-level
0.05	0.03274	0.06547
0.10	0.1290	0.2580
0.15	0.2832	0.5664
0.20	0.4863	0.9727
0.25	0.7268	1.454
0.30	0.9909	1.982
0.35	1.264	2.528
0.40	1.532	3.065
0.45	1.781	3.563
0.50	2.000	2.000
0.55	2.179	3.660
0.60	2.312	3.495
0.65	2.396	3.534
0.70	2.432	3.733
0.75	2.424	3.717
0.80	2.378	3.844
0.85	2.303	4.077
0.90	2.208	4.224
0.95	2.104	4.181
1.00	1.000	2.000

Table 2. The normalized squared free Euclidean distance, $d_{\min}^2(h)$, for uncoded binary and 4-level CPFSK modulation. See also fig. 47 and fig. 66.

Coded binary CPFSK modulation				
h	v=1		v=2	
	encoder, N	$d_{\min}^2(h)$	encoder, N	$d_{\min}^2(h)$
0.05	{(1,2),6}	0.1290	{(4,3),22}	0.1144
0.10	{ " " }	0.4863	{ " " }	0.4490
0.15	{ " " }	0.9909	{ " " }	0.9786
0.20	{ " " }	1.532	{ " " }	1.664
0.25	{ " " }	2.000	{ " " }	2.454
0.30	{ " 8 }	2.312	{ " " }	3.291
0.35	{(2,1),8}	2.852	{ " 26 }	4.116
0.40	{ " 10 }	3.341	{ " 22 }	4.874
0.45	{ " 12 }	3.732	{ " 24 }	5.514
0.50	{(2,3),12}	4.000	{ " 20 }	6.000
0.55	{(2,1),18}	4.130	{ " " }	5.592
0.60	{ " 10 }	4.121	{ " " }	5.366
0.65	{ " " }	3.984	{ " 54 }	4.943
0.70	{ " " }	3.741	{ " 16 }	4.484
0.75	{ " 8 }	3.000	{ " 14 }	4.000
- " -	{(2,3),8}	- " -		
0.80	{(2,1),8}	3.069	{(4,5),14}	3.652
0.85	{ " " }	2.715	{(5,7),14}	3.536
0.90	{ " " }	2.399	{ " 16 }	3.368
0.95	{ " 16 }	2.152	{ " 30 }	3.175
1.00	{(2,3),4}	1.500	{ " 12 }	2.500
- " -	{(1,3),6}	- " -		

Table 3. The optimal rate 1/2 convolutional encoders and the corresponding $d_{\min}^2(h)$ for different v and h. See also fig. 47.

Coded binary CPFSK modulation				
h	$v=3$		$v=4$	
	encoder, N	$d_{\min}^2(h)$	encoder, N	$d_{\min}^2(h)$
0.05	{(2, 11), 26}, {(12, 7), 32}	0.1225	{(31, 17), 34}, {(32, 17), 36}	0.1553
0.10	{ " " }, { " " }	0.4800	{ " " }, { " " }	0.6090
0.15	{ " " }, { " " }	1.043	{ " " }, { " " }	1.326
0.20	{ " " }, { " " }	1.766	{ " " }, { " " }	2.252
0.25	{ " " }, { " " }	2.590	{ " " }, { " " }	3.317
0.30	{ " " }, { " " }	3.450	{ " " }, { " " }	4.441
0.35	{ " 28 }, { " " }	4.278	{(22, 13), 54}, {(34, 17), 78}	5.380
0.40	{(12, 7), 30}	5.012	{ " 32 }	5.905
0.45	{ " " }	5.599	{(30, 13), 36}	6.950
0.50	{(1, 14), 22}, {(1, 17), 22}	6.000	{(22, 13), 32}, {(22, 15), 34}	8.000
-"	{(2, 11), 22}, {(2, 17), 22}	-"	{(22, 31), 32}, {(30, 13), 34}	-"
-"	{(3, 15), 20}, {(4, 17), 22}	-"	{(31, 17), 30}, {(32, 17), 34}	-"
-"	{(6, 13), 24}, {(10, 3), 22}	-"	{(34, 17), 32}	-"
-"	{(10, 5), 22}, {(12, 7), 26}	-"		
-"	{(12, 15), 26}, {(14, 7), 20}	-"		
-"	{(14, 13), 24}	-"		
0.55	{(10, 3), 24}	5.923	{(32, 17), 32}	7.422
0.60	{ " " }	5.930	{(22, 15), 28}	6.344
0.65	{ " 22 }	5.572	-----	---
0.70	{ " 20 }	5.051	{(24, 7), 26}	6.174
0.75	{(14, 7), 20}	5.000	{ " 24 }, {(31, 17), 22}	5.712
-"			{(20, 35), 24}, {(32, 17), 24}	-"
-"			{(24, 37), 28}, {(30, 13), 26}	-"
-"			{(34, 27), 26}	-"
0.80	{(13, 15), 18}	4.663	{(22, 31), 24}	5.295
0.85	{ " 20 }	4.606	-----	---
0.90	{ " 22 }	4.416	{(23, 35), 26}	4.679
0.95	{ " 30 }	4.207	-----	---
1.00	{(7, 13), 20}, {(7, 15), 22}	3.000	-----	---
-"	{(13, 7), 20}, {(13, 15), 20}	-"		
-"	{(13, 17), 16}, {(15, 7), 22}	-"		
-"	{(15, 13), 20}, {(15, 17), 18}	-"		

Table 4. The optimal rate 1/2 convolutional encoders and the corresponding $d_{\min}^2(h)$ for different v and h . See also fig. 47.

N	h=0.05 (31,17)	h=0.10 (31,17)	h=0.15 (31,17)	h=0.20 (31,17)	h=0.25 (31,17)	h=0.30 (31,17)
4	0.04902	0.1923	0.4185	0.7103	1.045	1.398
6	0.06531	0.2555	0.5538	0.9342	1.363	1.804
8	0.07350	0.2877	0.6246	1.056	1.545	2.052
10	0.07358	0.2890	0.6309	1.075	1.590	2.141
12	0.07358	0.2890	0.6309	1.075	1.590	2.141
14	0.08176	0.3213	0.7017	1.197	1.772	2.389
16	0.08995	0.3535	0.7725	1.318	1.954	2.636
18	0.09805	0.3845	0.8370	1.421	2.090	2.795
20	0.1062	0.4168	0.9078	1.542	2.272	3.043
22	0.1144	0.4490	0.9786	1.664	2.454	3.291
24	0.1226	0.4813	1.049	1.785	2.635	3.539
26	0.1227	0.4826	1.056	1.805	2.680	3.627
28	0.1309	0.5148	1.127	1.926	2.862	3.875
30	0.1391	0.5471	1.197	2.048	3.044	4.123
32	0.1472	0.5780	1.262	2.150	3.180	4.282
34	0.1553	0.6090	1.326	2.252	3.317	4.441
	=====	=====	=====	=====	=====	=====

Table 5. The minimum normalized squared Euclidean distance growth, $d_{\min, N}^2(h)$, for the (31,17)-encoder ($v=4$). Binary CPFSK modulation is assumed.

N	h=0.35 (22,13)	h=0.40 (22,13)	h=0.45 (30,13)	h=0.50 (31,17)	h=0.55 (32,17)
4	1.742	2.054	2.312	2.500	2.261
6	2.220	2.458	2.681	3.000	3.001
8	2.536	2.926	2.757	3.500	3.467
10	2.690	3.203	2.757	4.000	3.933
12	3.006	3.586	2.757	4.000	4.382
14	3.322	3.872	3.202	4.500	4.756
16	3.322	3.969	3.648	5.000	4.903
18	3.638	4.352	3.648	5.000	4.903
20	3.749	4.735	4.093	5.500	5.409
22	3.987	5.118	4.538	6.000	5.954
24	4.225	5.348	4.984	6.500	6.245
26	4.463	5.501	5.429	7.000	6.777
28	4.587	5.501	5.874	7.500	6.924
30	4.750	5.864	6.233	8.000	6.924
32	4.843	5.905	6.581	=====	7.422
34	4.891	=====	6.930	=====	=====
36	4.940		6.950		
38	4.989		=====		
40	5.038				
42	5.087				
44	5.136				
46	5.185				
48	5.234				
50	5.283				
52	5.332				
54	5.380				
	=====				

Table 6. The minimum normalized squared Euclidean distance growth, $d_{\min, N}^2(h)$, for some selected $v=4$ encoders. Binary CPFSK modulation is assumed.

Coded 4-level CPFSK modulation				
h	v=1		v=2	
	enc., map., N	$d_{\min}^2$ (h)	encoder, mapping rule, N	$d_{\min}^2$ (h)
0.05	{(1,2),Q1,4}	0.1935	{(7,2),Q1,13},{(7,5),Q2,13}	0.2587
0.10	{ " " " }	0.7295	{ " " " },{ " " " }	0.9826
0.15	{ " " " }	1.486	{ " " " },{ " " " }	2.029
0.20	{ " " " 5}	2.298	{ " " " 12},{ " " " 12}	3.202
0.25	{ " " " 6}	3.000	{ " " " 10},{ " " " 10}	4.302
0.30	{ " " " 11}	3.468	{ " " " 14},{ " " " 14}	4.312
0.35	{ " " " 36}	3.389	{(3,4),Q1,40}	4.164
0.40	{ " " " 5}	3.126	{(1,4),Q1,6},{(1,5),Q2,7}	3.911
0.45	{ " " " 7}	3.312	{(1,7),Q1,8},{(1,6),Q2,8}	4.296
0.50	{ " " " 3}	3.000	{ " " " 7},{ " " " 7}	5.000
-"	{(3,1),Q1,3}	-"	{(3,7),Q1,7}	-"
-"	{(1,3),Q2,3}	-"		-"
0.55	{ " " " 4}	2.977	{(1,7),Q1,7},{(1,6),Q2,7}	4.346
-"			{(3,7),Q1,7}	-"
0.60	{ " " " " }	2.998	{(1,4),Q1,6},{(1,5),Q2,6}	4.060

Table 7. The optimal combinations of a rate 1/2 convolutional encoder and a 4-level mapping rule, and the corresponding $d_{\min}^2$ (h) for v=1 and 2. See also fig. 66.

Coded 4-level CPFSK modulation			
h	v=3. Encoder, mapping rule, N		$d_{\min}^2$ (h)
0.05	{(13,2),Q1,20},{(13,11),Q2,20},{(13,4),Q1,26},{(13,17),Q2,26}		0.3075
0.10	{ " " " " },{ " " " " },{ " " " " },{ " " " " }		1.171
0.15	{ " " " " },{ " " " " },{ " " 23},{ " " 23}		2.430
0.20	{ " " " " },{ " " " " }		3.863
0.25	{ " " 22},{ " " 22}		5.241
0.30	{(3,10),Q1,20}		5.080
0.35	{ " " 26}		5.006
0.40	{ " " 13}		4.602
0.45	{(1,15),Q1,12},{(1,14),Q2,12}		5.514
0.50	{ " " 9},{ " " 9},{(2,15),Q1,10},{(2,17),Q2,10}		6.000
-"	{(3,13),Q1,9},{(3,15),Q1,9},{(3,16),Q1,8},{(7,13),Q1,9}		-"
-"	{(7,14),Q2,9},{(13,7),Q1,9},{(13,14),Q2,9},{(17,13),Q1,9}		-"
0.55	{(1,15),Q1,11},{(1,14),Q2,11},{(2,15),Q1,10},{(2,17),Q2,10}		5.611
-"	{(3,13),Q1,11}		-"
0.60	{(3,10),Q1,9}		4.751

Table 8. The optimal combinations of a rate 1/2 convolutional encoder and a 4-level mapping rule, and the corresponding $d_{\min}^2$ (h) for v=3. See also fig. 66.

Coded 4-level CPFSK modulation. $v=4$		
h	enc., map., N	$d_{\min}^2(h)$
0.10 *	{(23,10), Q1, 27}	1.362
0.20 *	{ " " " 24}	4.427
0.25	{ " " " }	6.151
- "-	{(23,33), Q2, 24}	- "-
0.30 *	{(7,20), Q1, 19}	5.702
0.35 **	{(4,23) " 30}	5.380
0.40 *	{ " " " 15}	5.881
0.45 *	{ " " " 14}	6.760
0.50 **	{ " " " 13}	7.000
0.55 **	{ " " " }	6.696
0.60 *	{ " " " 11}	5.761

Table 9. The optimal combinations of a rate 1/2 convolutional encoder and a 4-level mapping rule, and the corresponding $d_{\min}^2(h)$ for $v=4$. See also fig. 66.

*) Search not complete, only the combinations

$\{\{g_{m,1}(i)\}, Q1\}$ have been searched.

**) Search not complete.

N	h				
	0.05	0.10	0.15	0.20	0.25
1	0.06451	0.2432	0.4954	0.7661	1.000
2	0.1130	0.4268	0.8730	1.358	1.788
3	0.1293	0.4913	1.015	1.601	2.151
4	0.1293	0.4913	1.015	1.601	2.151
5	0.1293	0.4913	1.015	1.601	2.151
6	0.1457	0.5558	1.156	1.844	2.515
7	0.1621	0.6203	1.298	2.087	2.878
8	0.1621	0.6203	1.298	2.087	2.878
9	0.1621	0.6203	1.298	2.087	2.878
10	0.1784	0.6848	1.439	2.331	3.241
11	0.1948	0.7493	1.581	2.574	3.605
12	0.1948	0.7493	1.581	2.574	3.605
13	0.1948	0.7493	1.581	2.574	3.605
14	0.2112	0.8139	1.723	2.817	3.968
15	0.2276	0.8784	1.864	3.060	4.331
16	0.2276	0.8784	1.864	3.060	4.331
17	0.2276	0.8784	1.864	3.060	4.331
18	0.2439	0.9429	2.006	3.303	4.695
19	0.2603	1.007	2.147	3.546	5.058
20	0.2603	1.007	2.147	3.546	5.058
21	0.2603	1.007	2.147	3.546	5.058
22	0.2767	1.072	2.289	3.790	<u>5.241</u>
23	0.2930	1.136	<u>2.430</u>	<u>3.863</u>	
24	0.2930	1.136			
25	0.2930	1.136			
26	<u>0.3075</u>	<u>1.171</u>			

Table 10. The minimum normalized squared Euclidean distance growth, $d_{\min, N}^2(h)$, when the combination $\{(13,4), Q1\}$ or $\{(13,17), Q2\}$ is used together with CPFSK modulation, for some selected modulation indices.

v	d_f best code	$d_{\min}^2(1/2)$: Optimal combination	
		M=2	M=4
1	3	4	3
2	5	6	5
3	6	6	6
4	7	8	7 **

Table 11. Comparison between the optimal rate 1/2 convolutional codes for $d_{\min}^2(1/2)$ and d_f .

**) Search not complete.

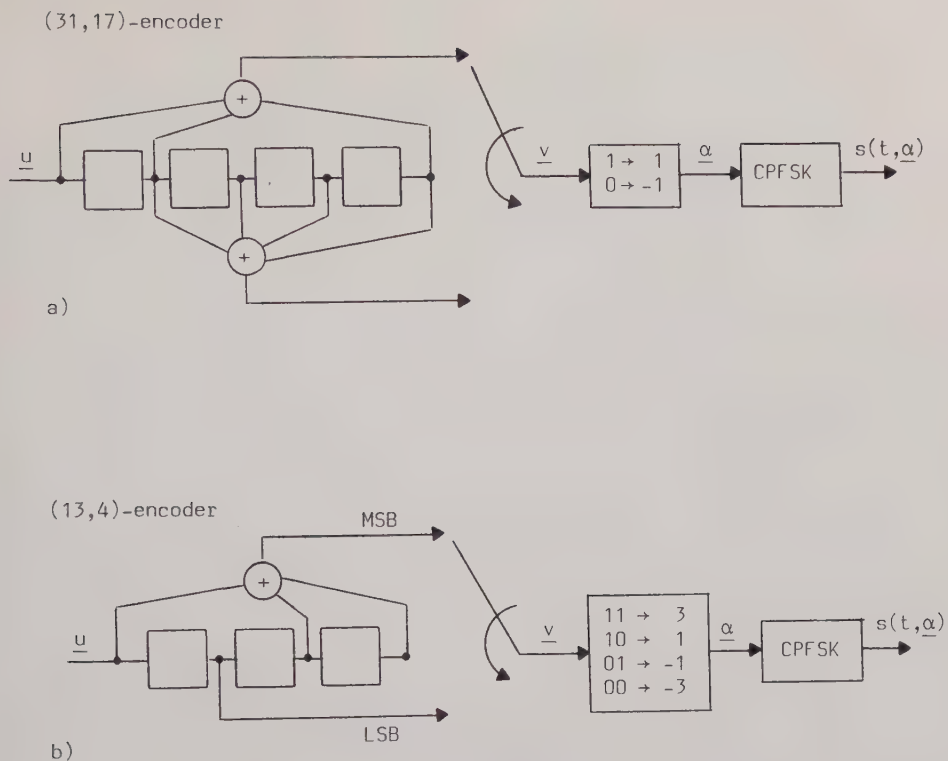
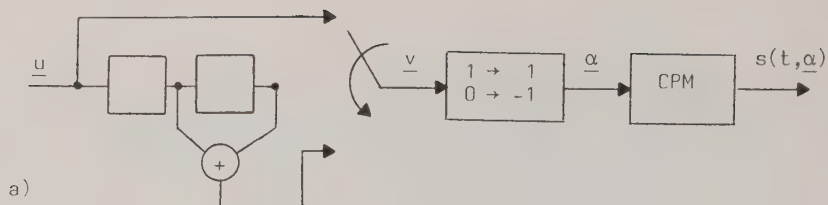


Figure 1. Examples of modulation schemes considered in this part.

- The rate $1/2$ (31,17)-encoder combined with binary CPFSK modulation.
- The rate $1/2$ (13,4)-encoder combined with 4-level (nat.bin.) CPFSK modulation.

(4,3)-encoder



(6,1)-encoder

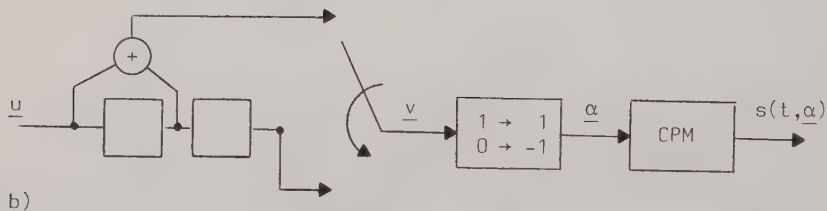


Figure 2. Examples of schemes having the same free Euclidean distance.

- The $(4,3)$ -encoder.
- The $(6,1)$ -encoder.

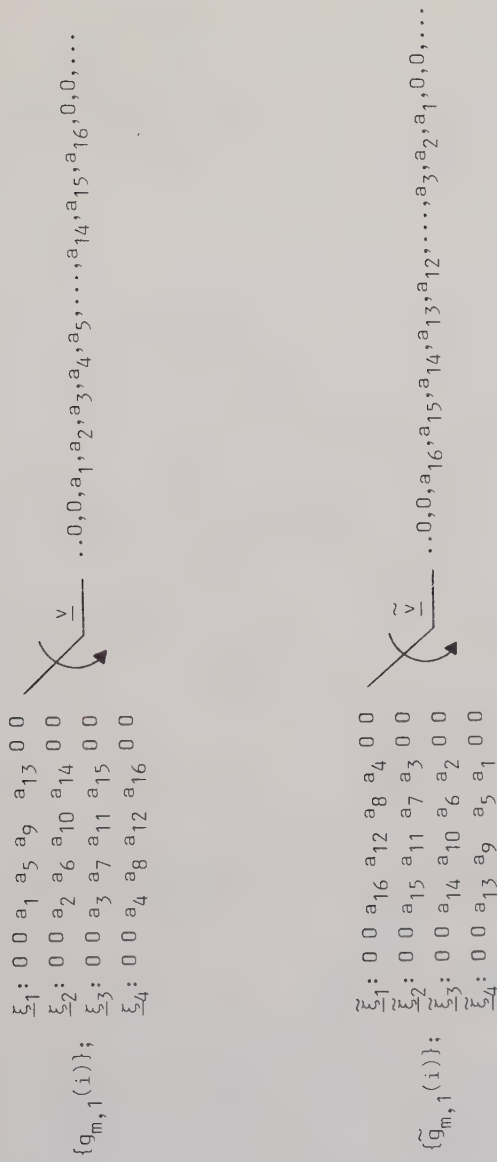


Figure 3. Illustrating the relationship between $\underline{\xi}_m$ and $\underline{\xi}_{n+1-m}$ and also between $\underline{v}$ and $\underline{\tilde{v}}$. $n=4$ in this figure. See section II for details.

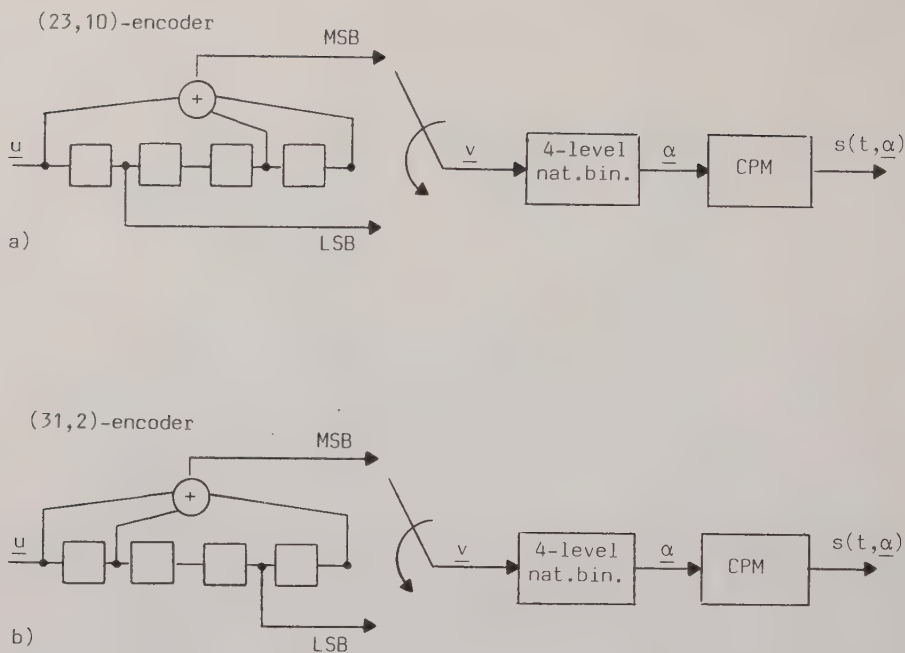


Figure 4. Examples of schemes having the same free Euclidean distance.

a) The $(23,10)$ -encoder.

b) The $(31,2)$ -encoder.

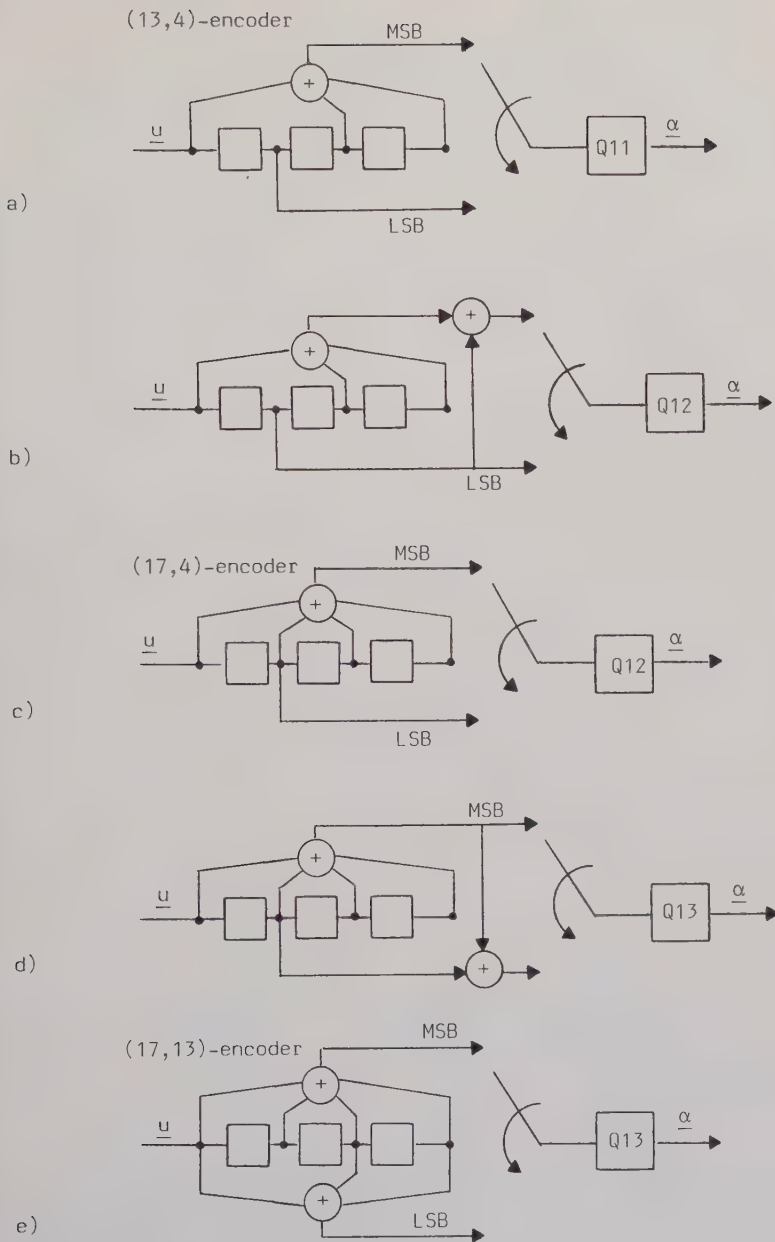


Figure 5. Illustrating how to obtain identical combinations of a rate $1/2$ convolutional encoder and a 4-level mapping rule. See section II for details.

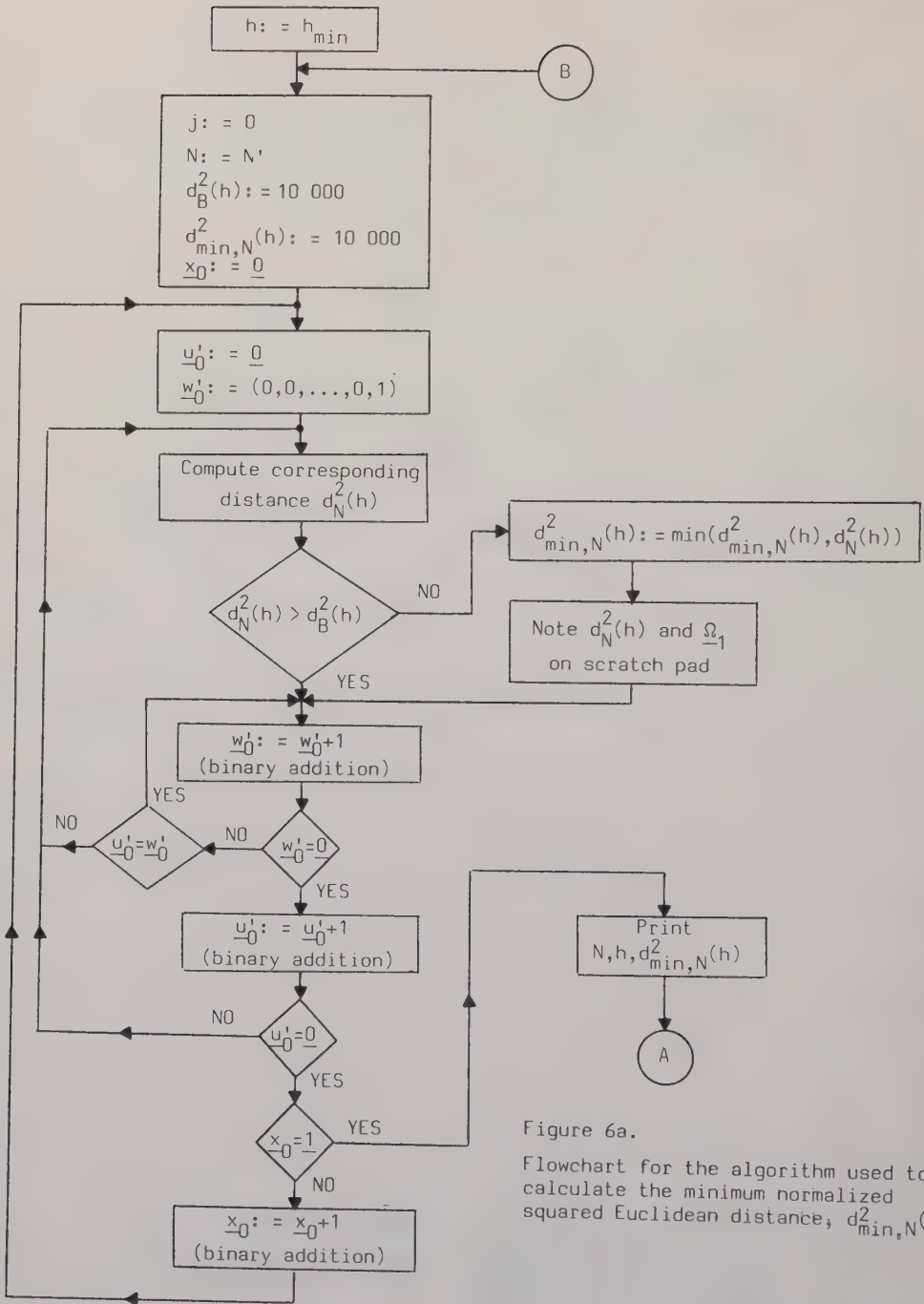


Figure 6a.

Flowchart for the algorithm used to calculate the minimum normalized squared Euclidean distance, $d_{\min,N}^2(h)$.

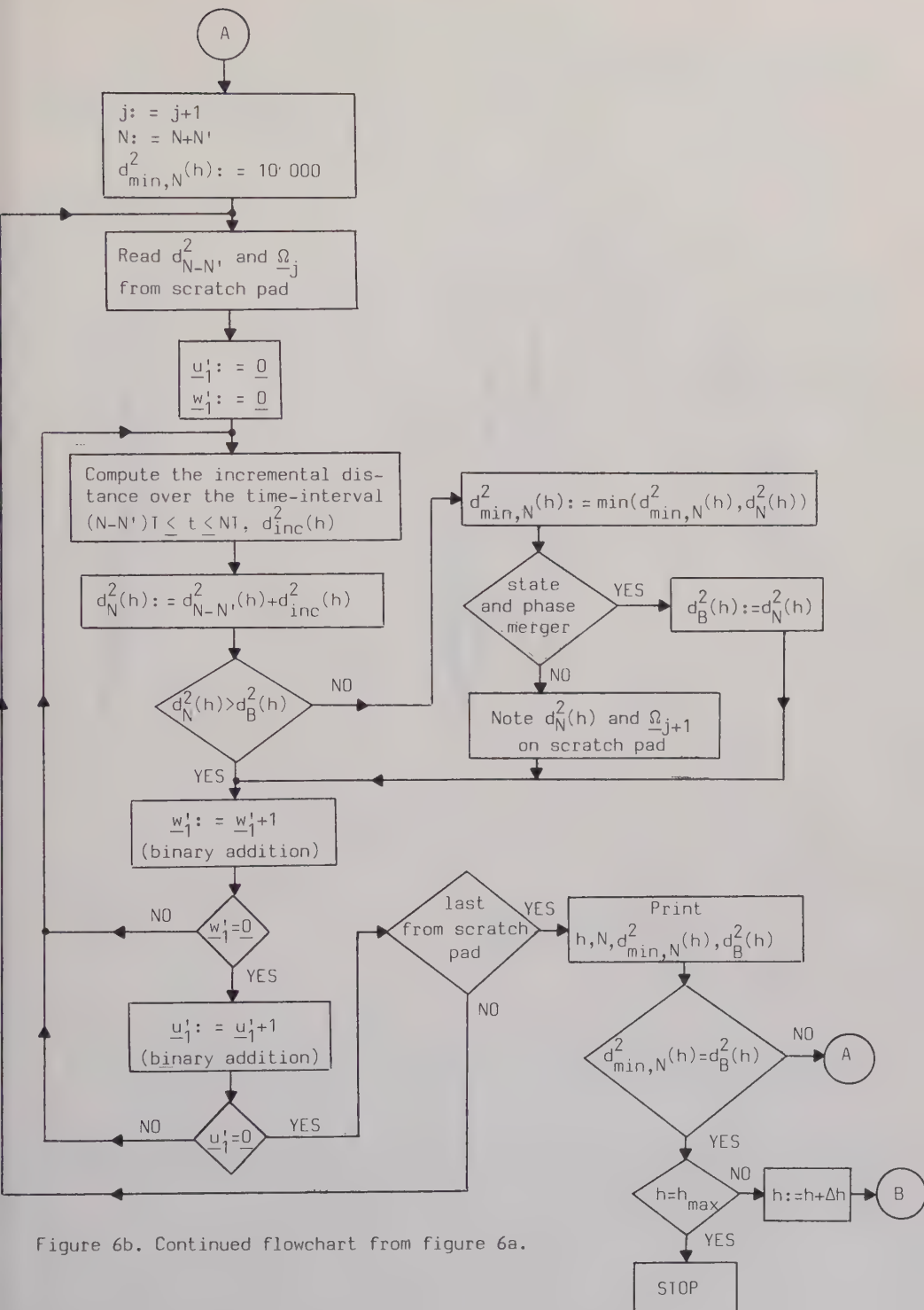


Figure 6b. Continued flowchart from figure 6a.

(1,2)-encoder

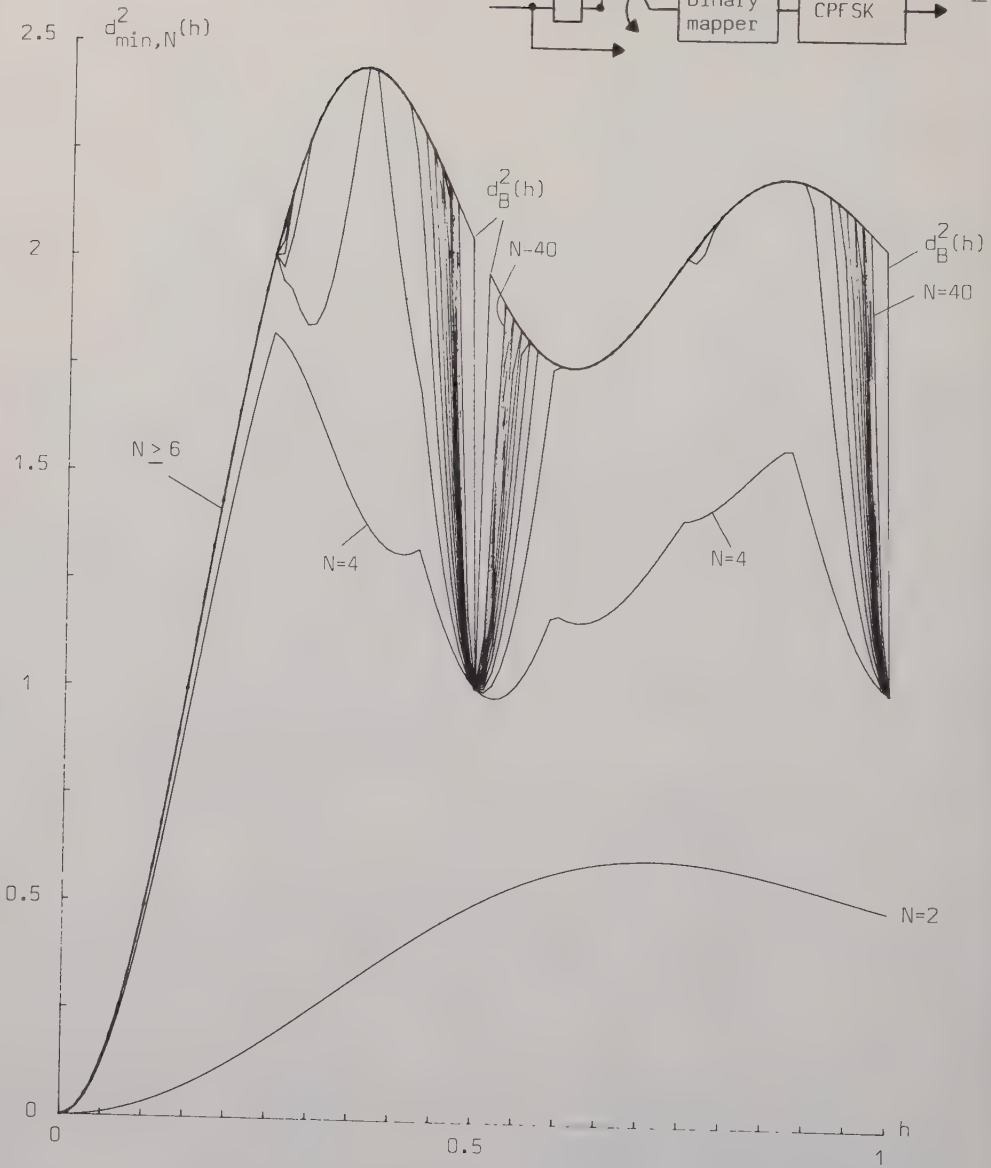
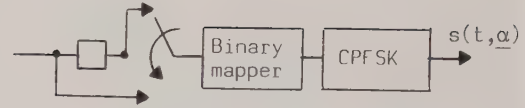


Figure 7. Minimum normalized squared Euclidean distance versus modulation index h when the (1,2)-encoder is used.

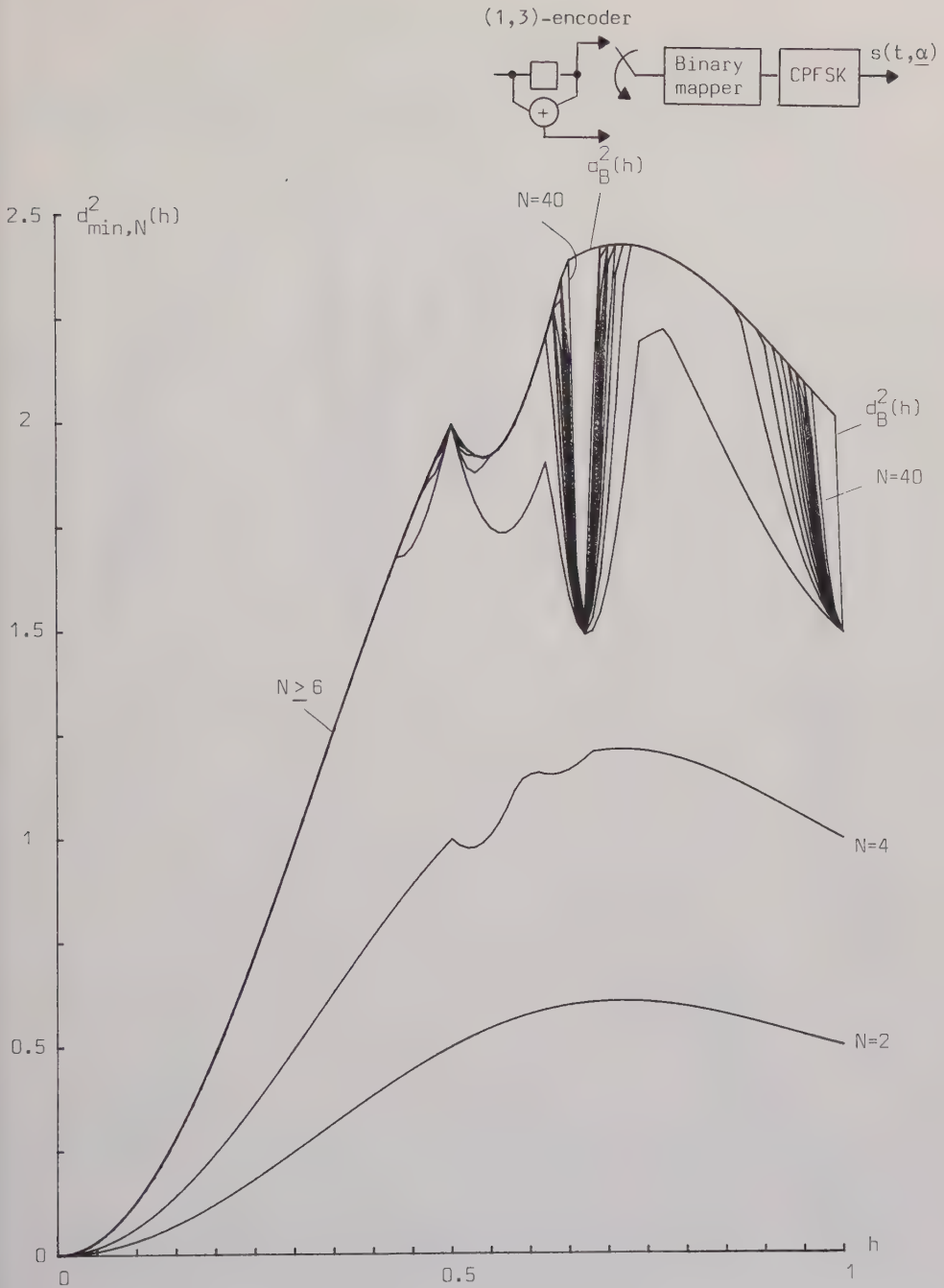


Figure 8. Minimum normalized squared Euclidean distance versus modulation index h when the (1,3)-encoder is used.

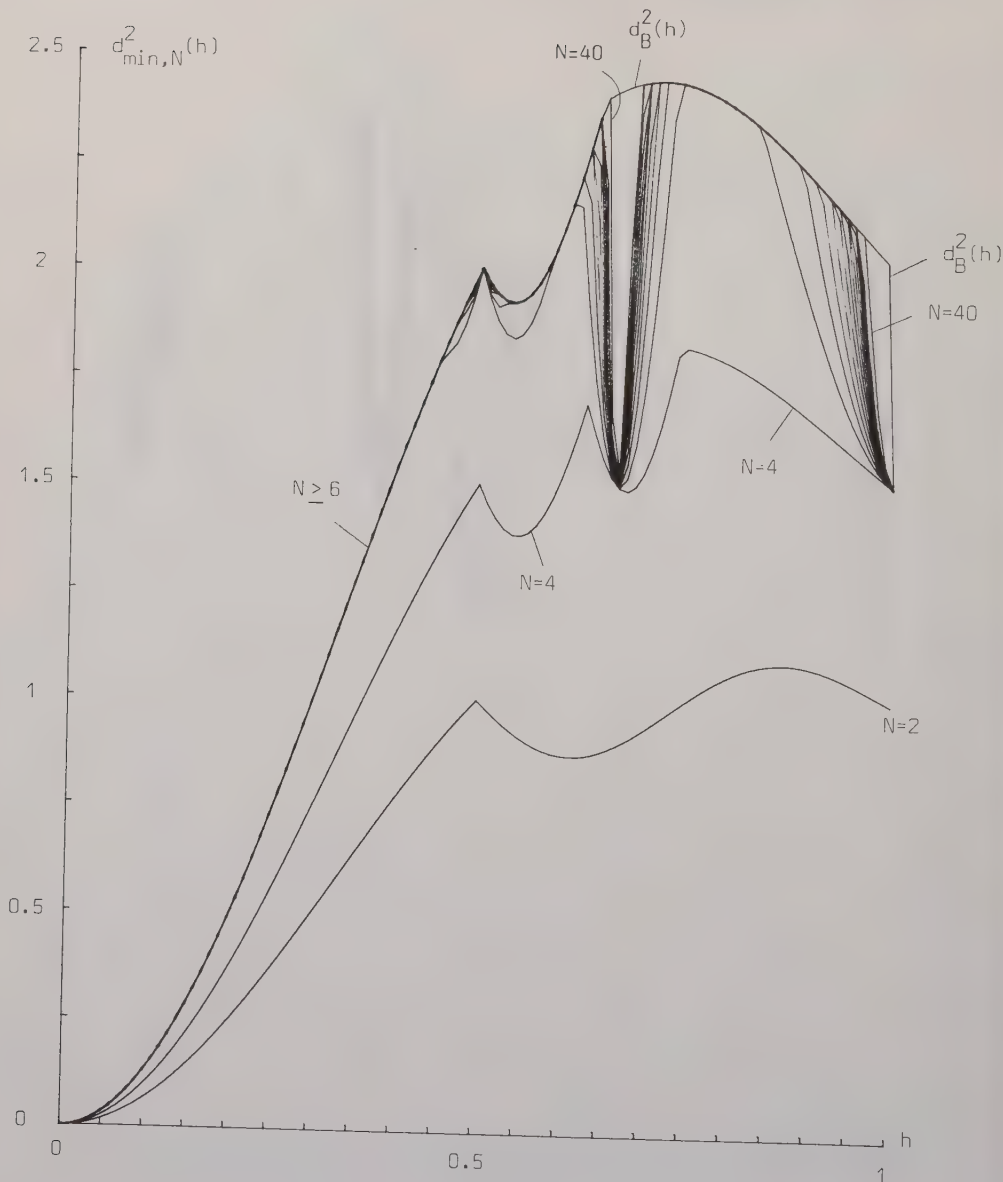
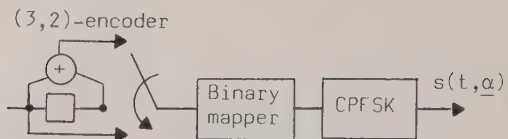


Figure 9. Minimum normalized squared Euclidean distance versus modulation index h when the (3,2)-encoder is used.

(2,1)-encoder

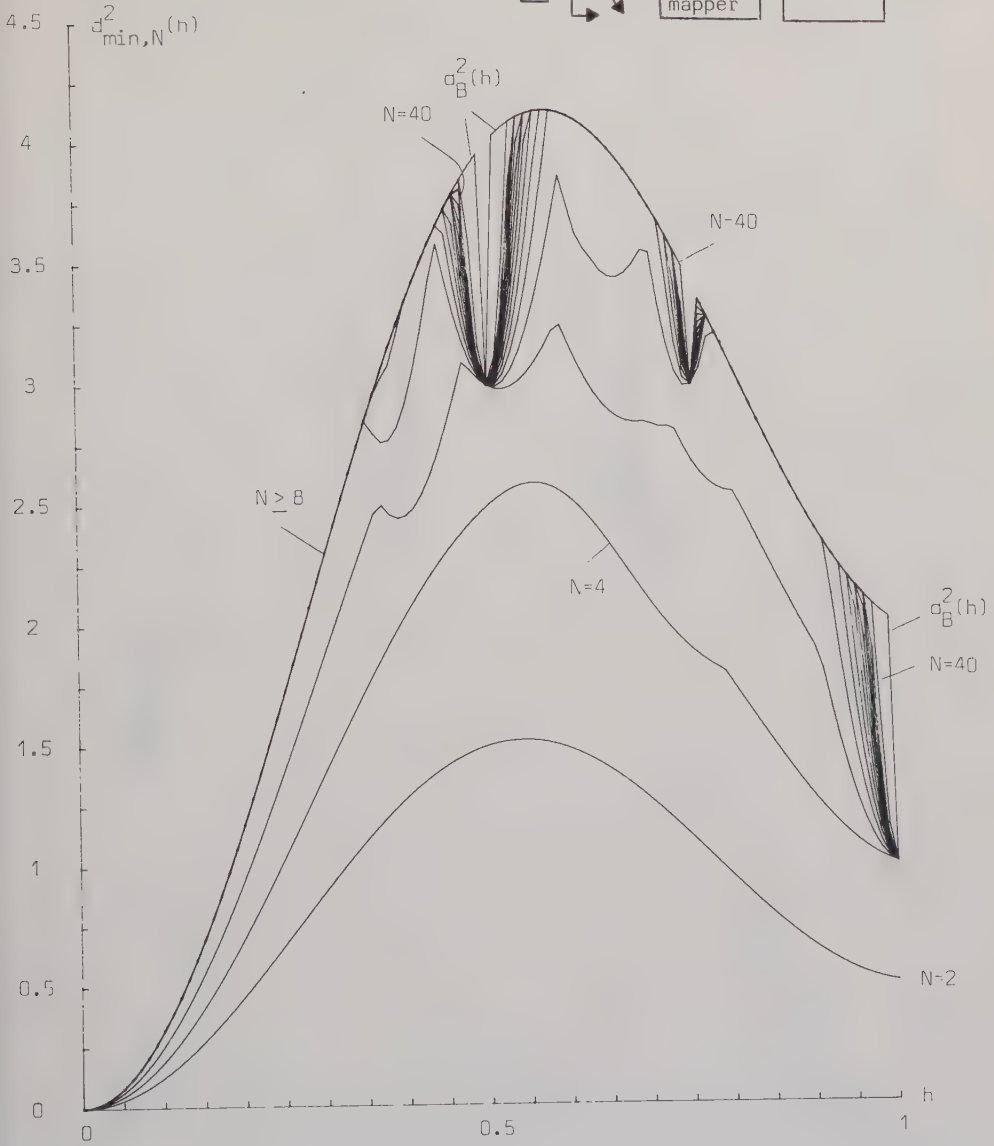
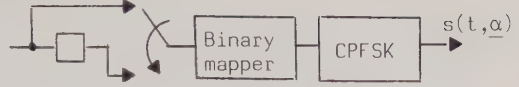


Figure 10. Minimum normalized squared Euclidean distance versus modulation index h when the (2,1)-encoder is used.

(2,3)-encoder

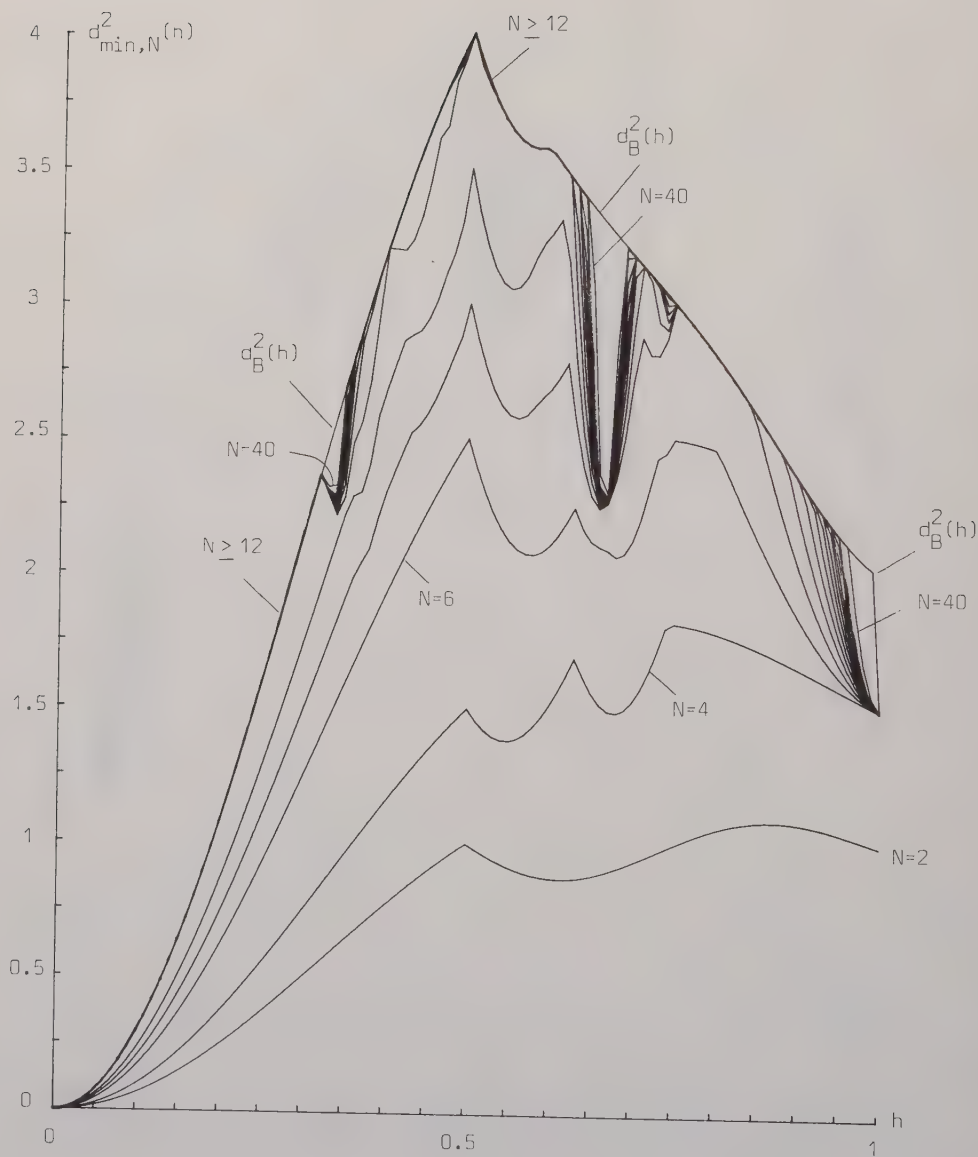
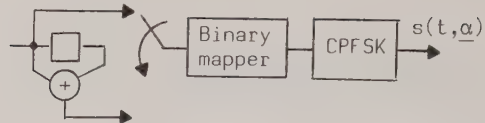


Figure 11. Minimum normalized squared Euclidean distance versus modulation index h when the (2,3)-encoder is used.

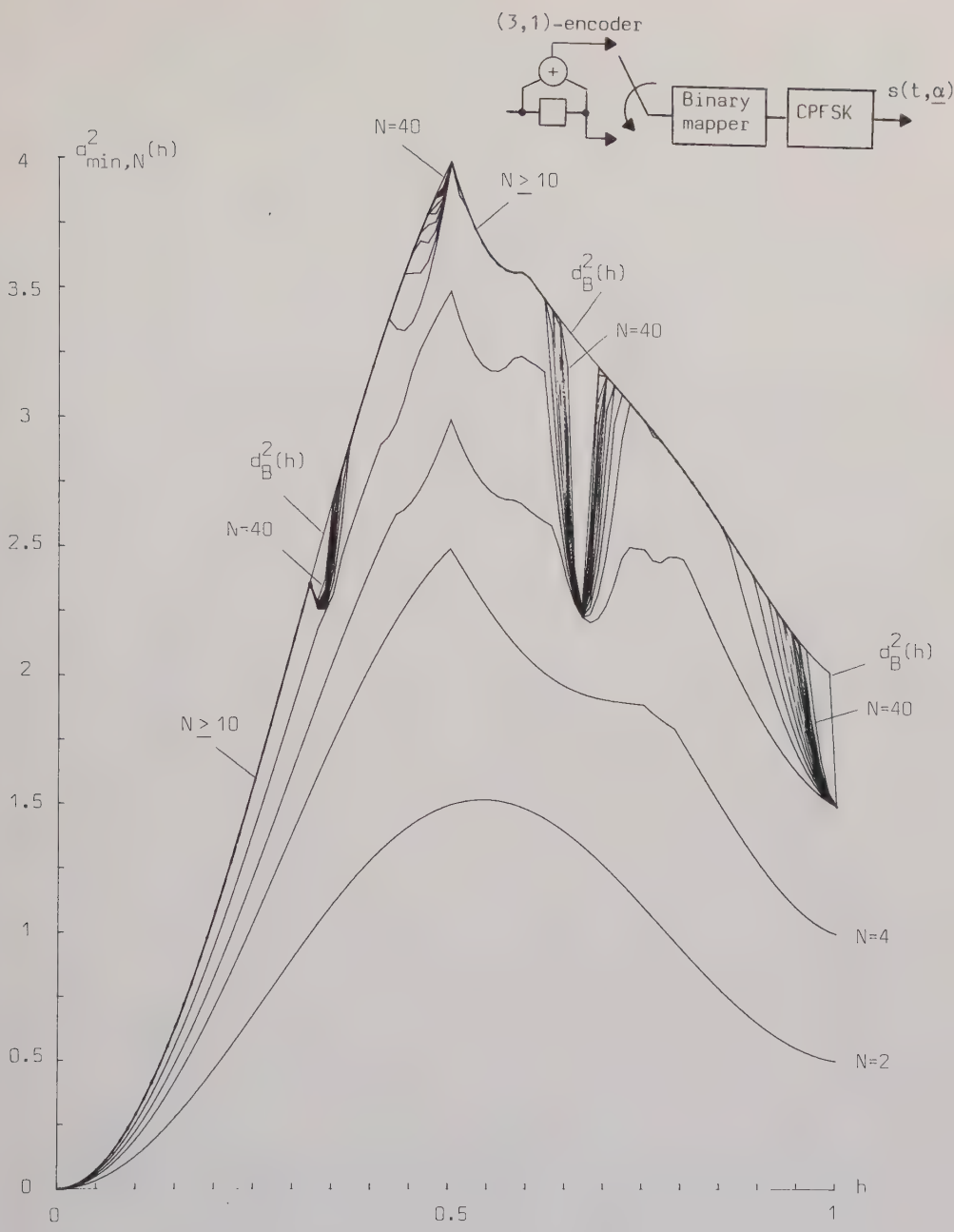


Figure 12. Minimum normalized squared Euclidean distance versus modulation index h when the $(3,1)$ -encoder is used.

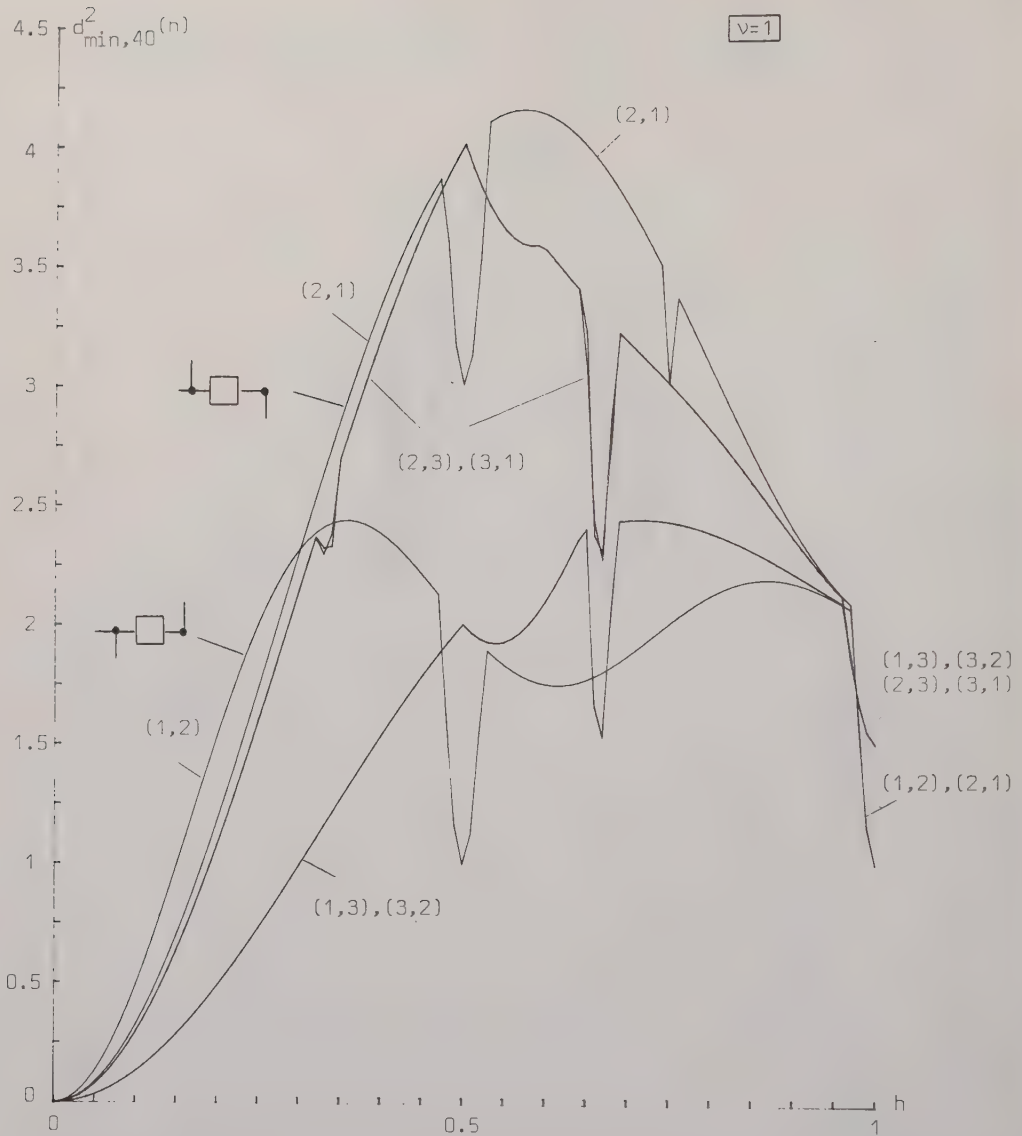


Figure 13. Minimum normalized squared Euclidean distance versus modulation index h for all noncatastrophic $v=1$ encoders, $N=40$.

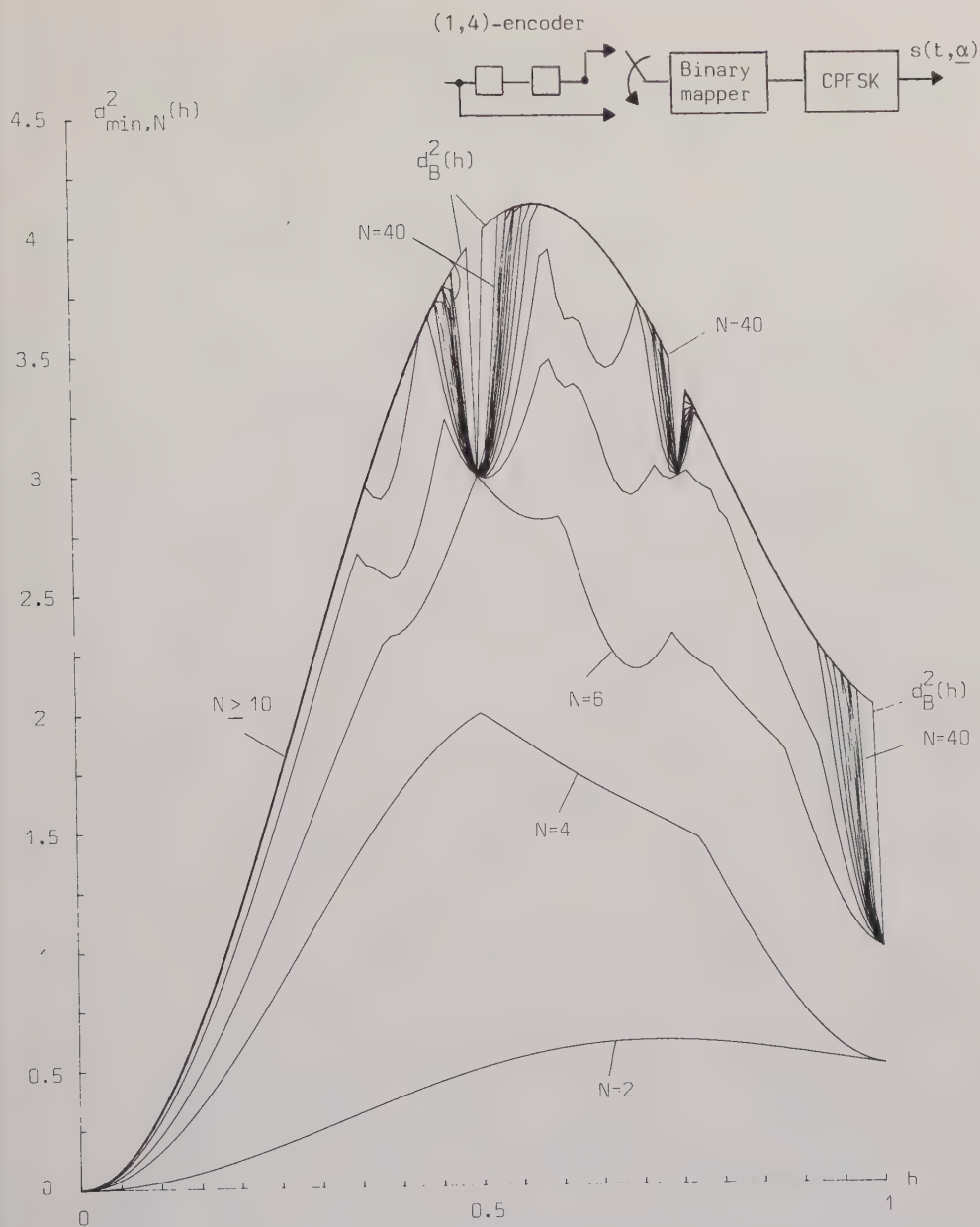


Figure 14. Minimum normalized squared Euclidean distance versus modulation index h when the (1,4)-encoder is used.

(1,5)-encoder

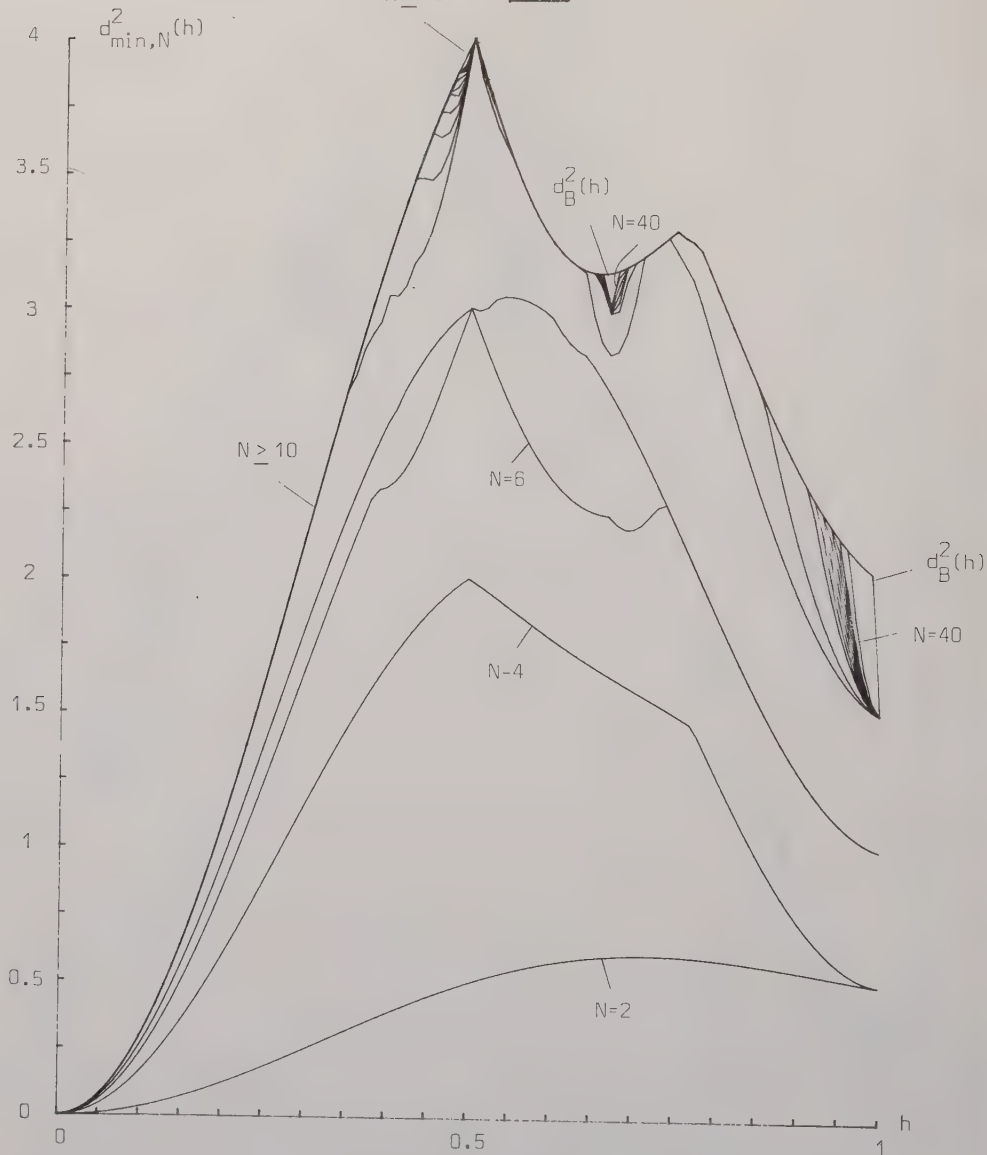
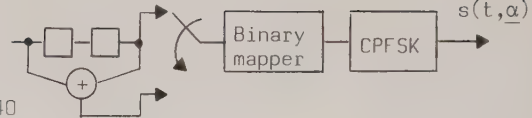


Figure 15. Minimum normalized squared Euclidean distance versus modulation index h when the (1,5)-encoder is used.

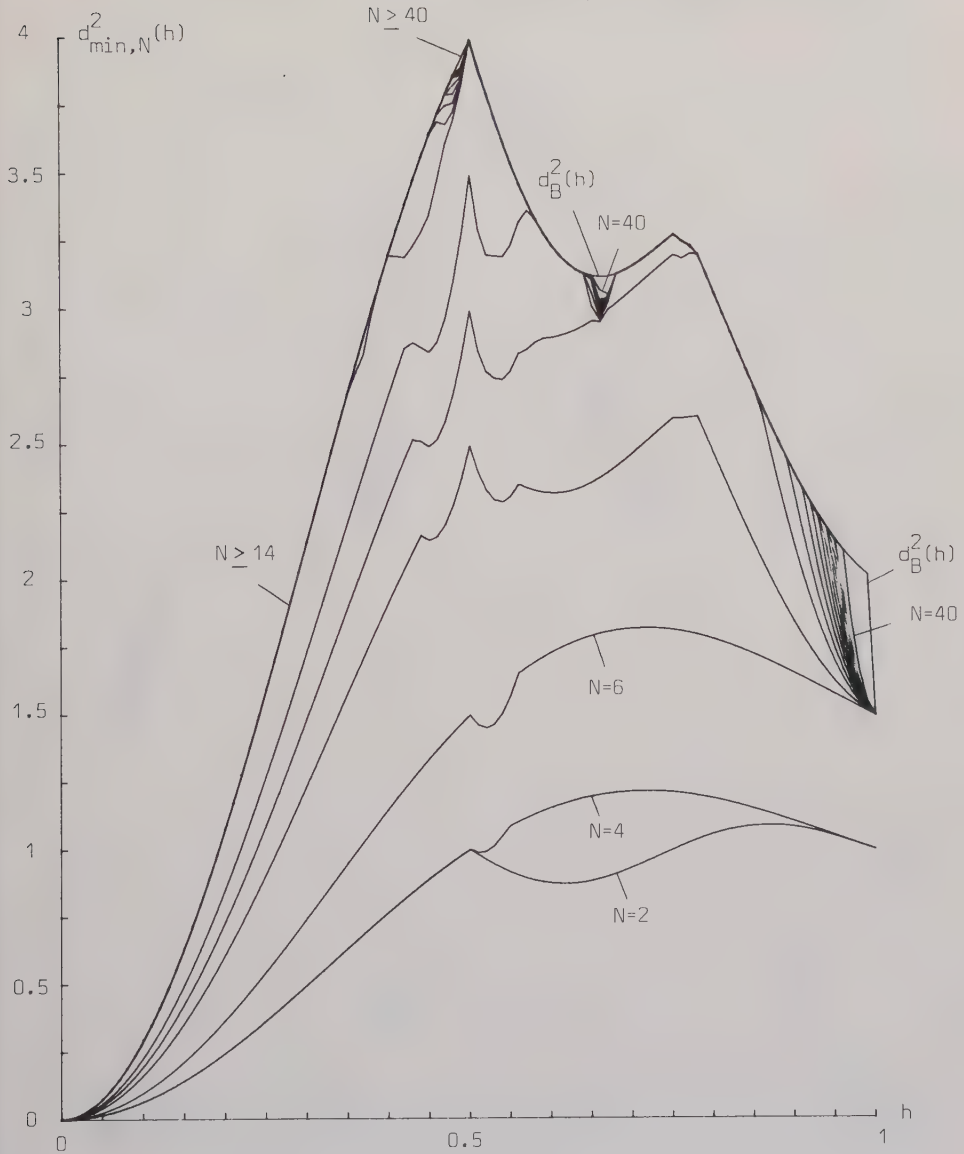
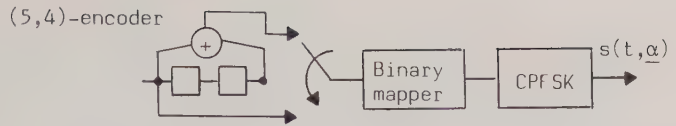


Figure 16. Minimum normalized squared Euclidean distance versus modulation index h when the (5,4)-encoder is used.

(1,6)-encoder

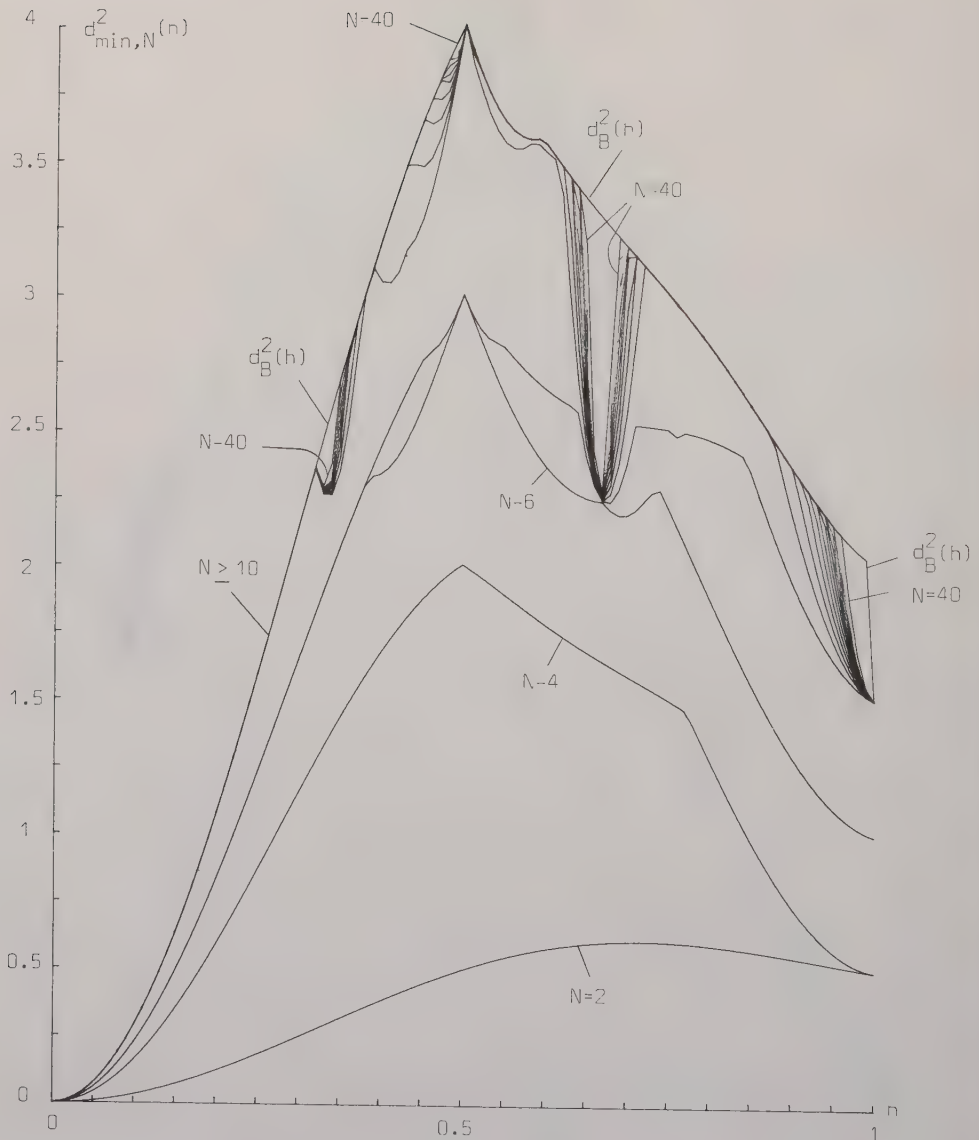
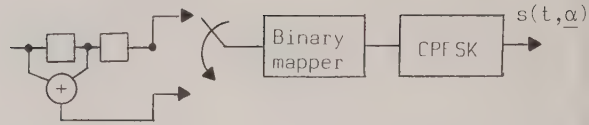


Figure 17. Minimum normalized squared Euclidean distance versus modulation index h when the (1,6)-encoder is used.

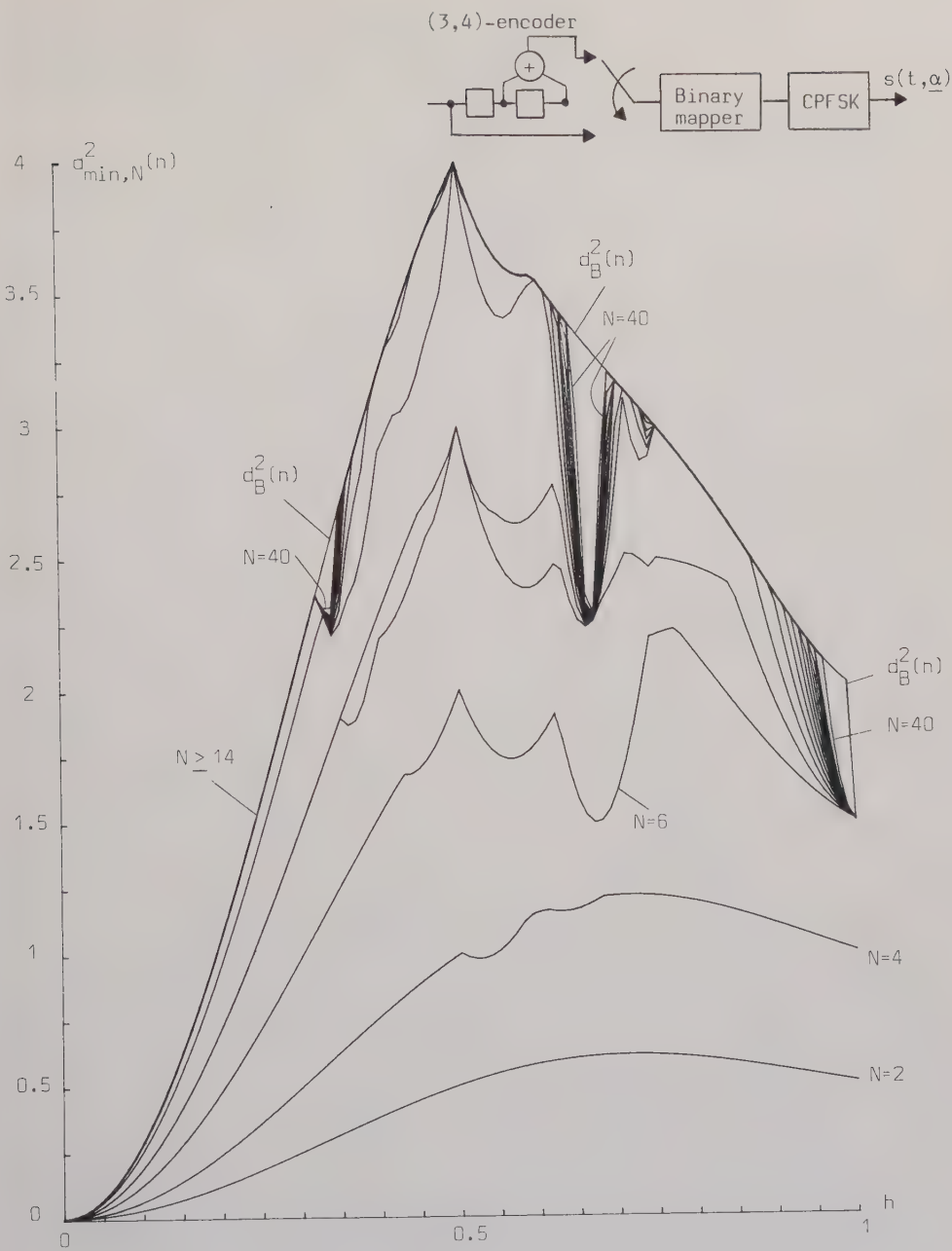


Figure 18. Minimum normalized squared Euclidean distance versus modulation index h when the (3,4)-encoder is used.

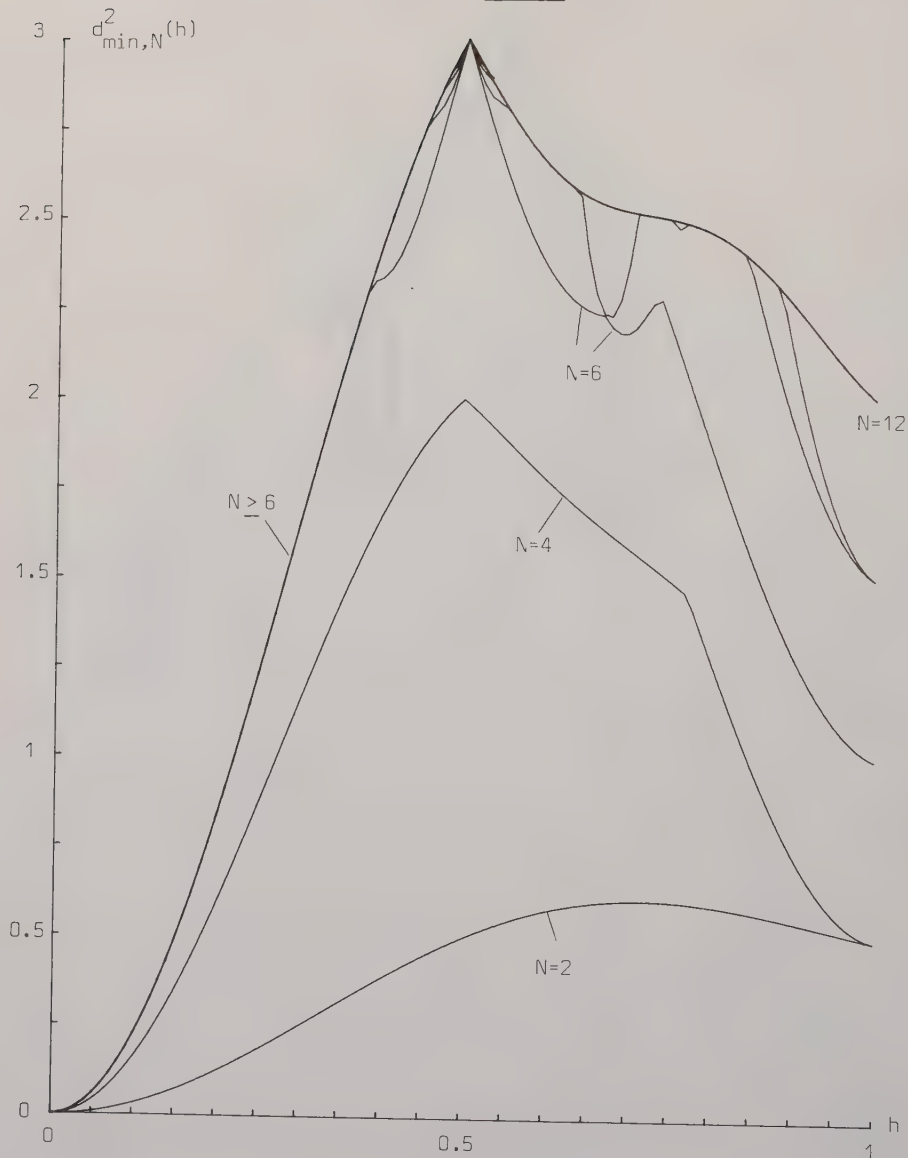
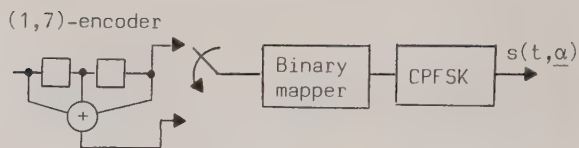


Figure 19. Minimum normalized squared Euclidean distance versus modulation index h when the (1,7)-encoder is used.

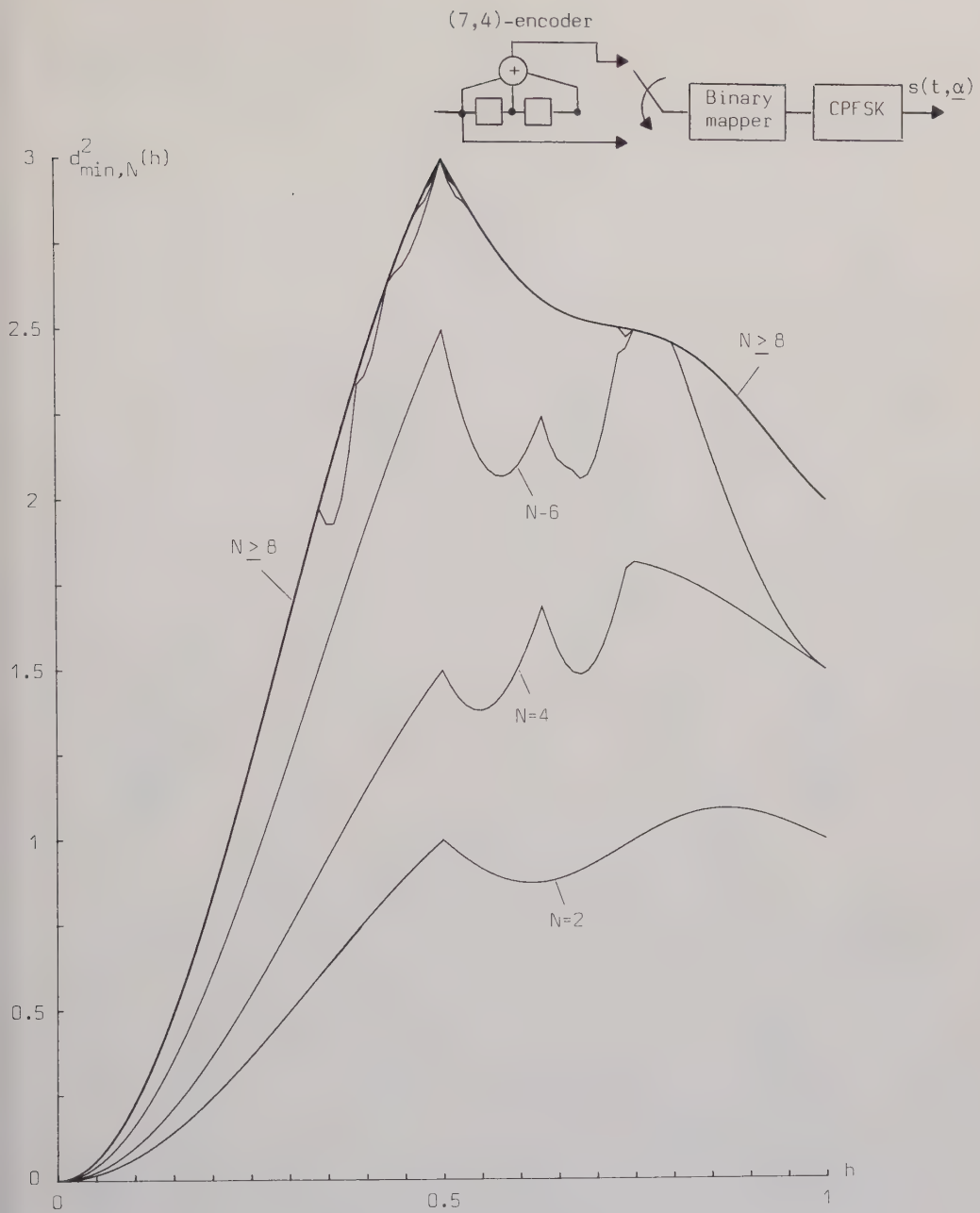
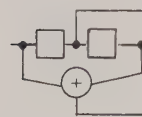


Figure 20. Minimum normalized squared Euclidean distance versus modulation index h when the (7,4)-encoder is used.

(2,5)-encoder



Binary mapper

CPFSK

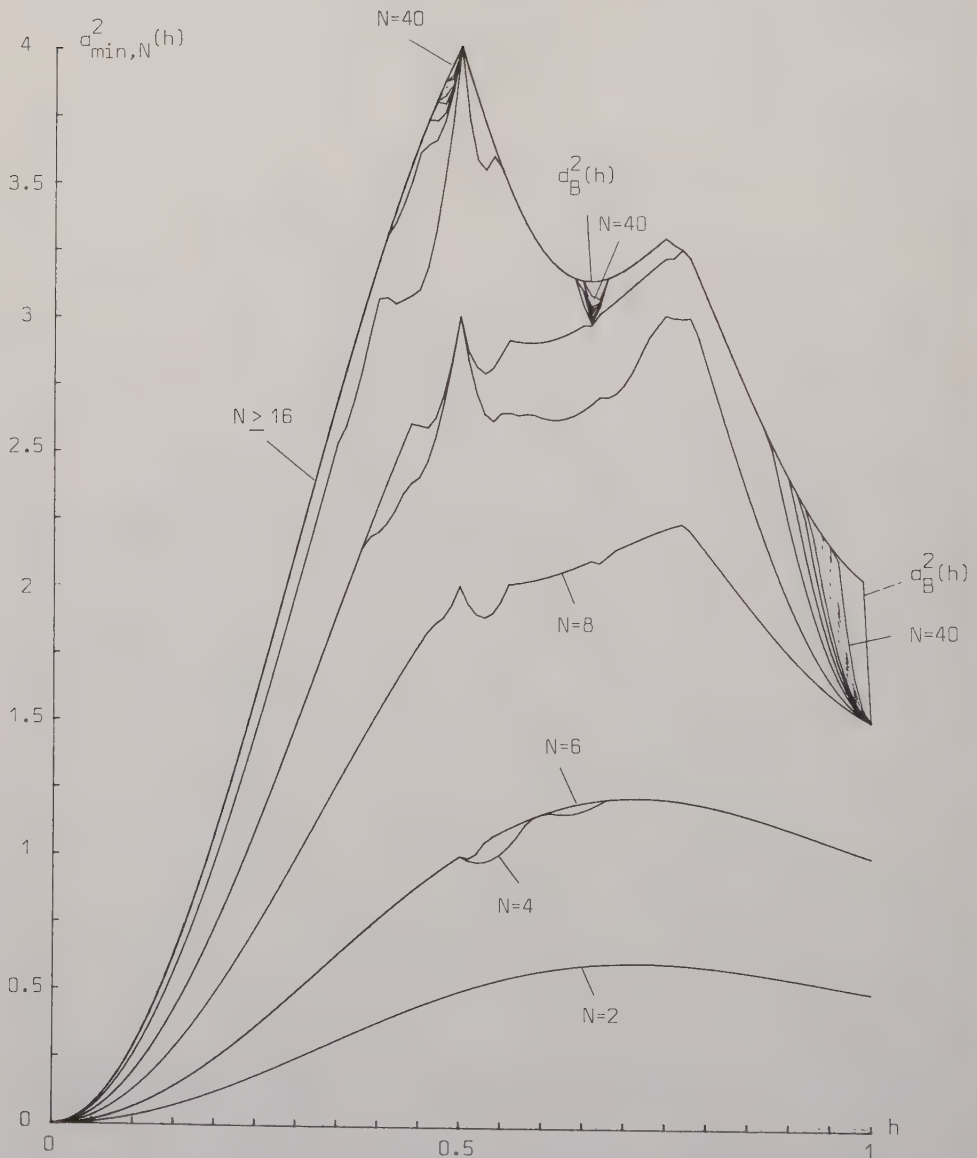
 $s(t, \underline{a})$ 

Figure 21. Minimum normalized squared Euclidean distance versus modulation index h when the (2,5)-encoder is used.

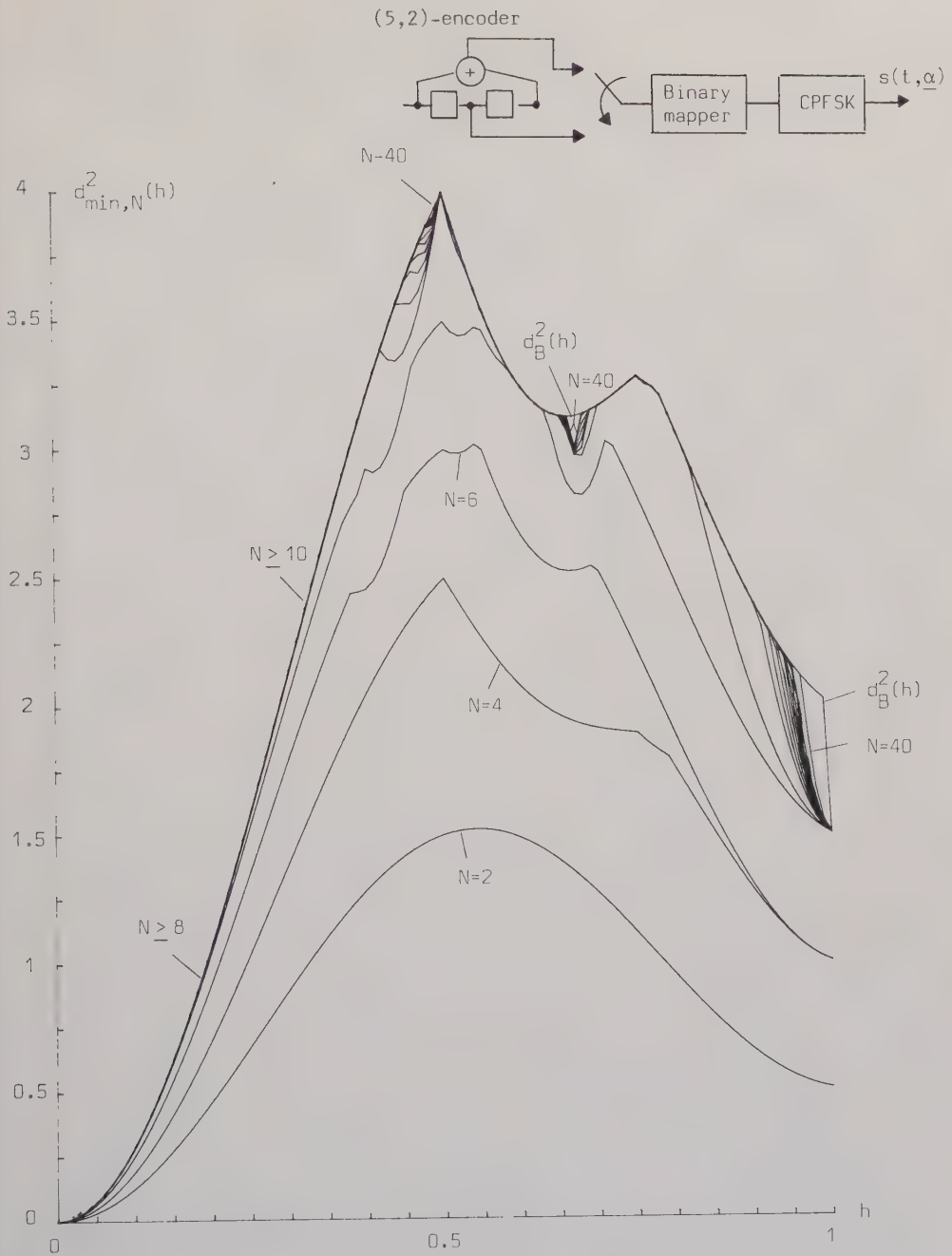


Figure 22. Minimum normalized squared Euclidean distance versus modulation index h when the (5,2)-encoder is used.

(2,7)-encoder

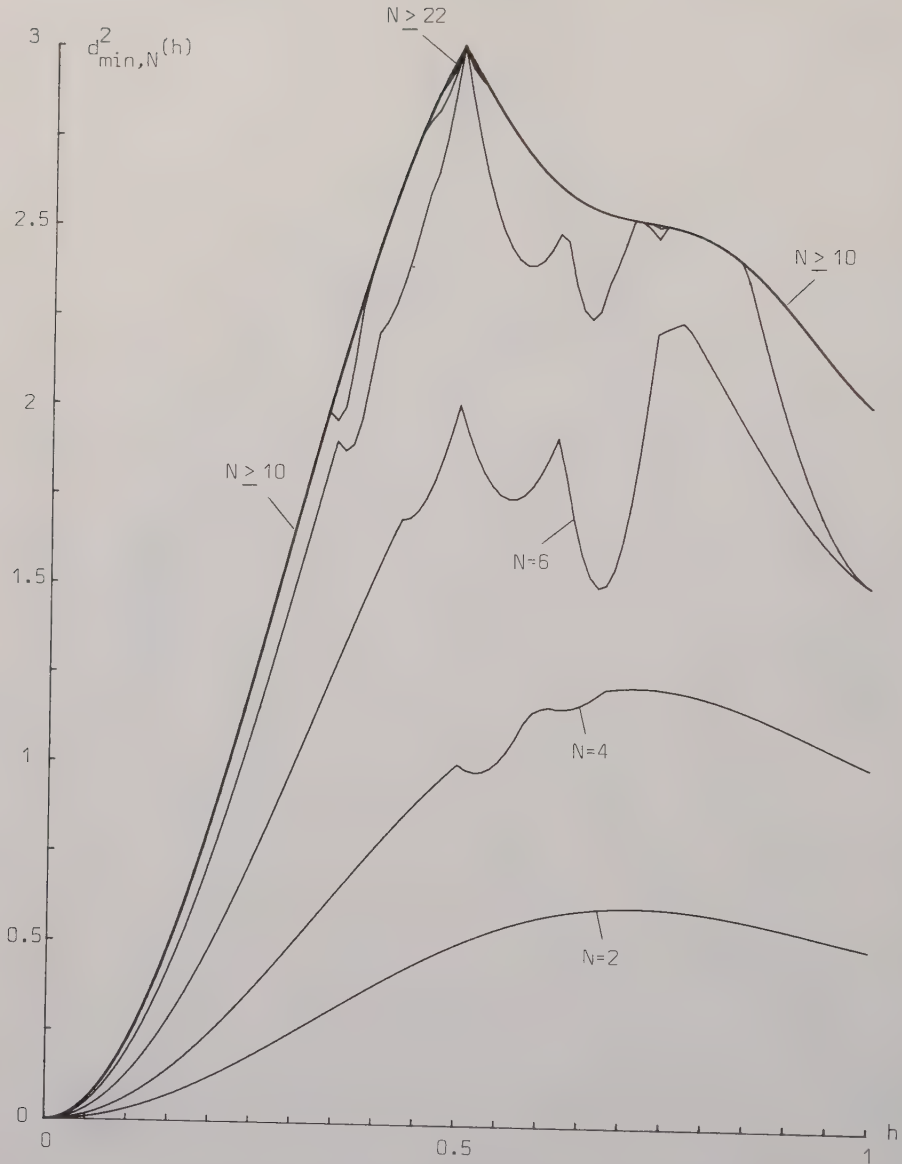
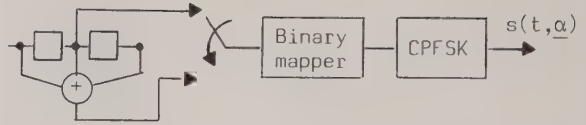


Figure 23. Minimum normalized squared Euclidean distance versus modulation index h when the (2,7)-encoder is used.

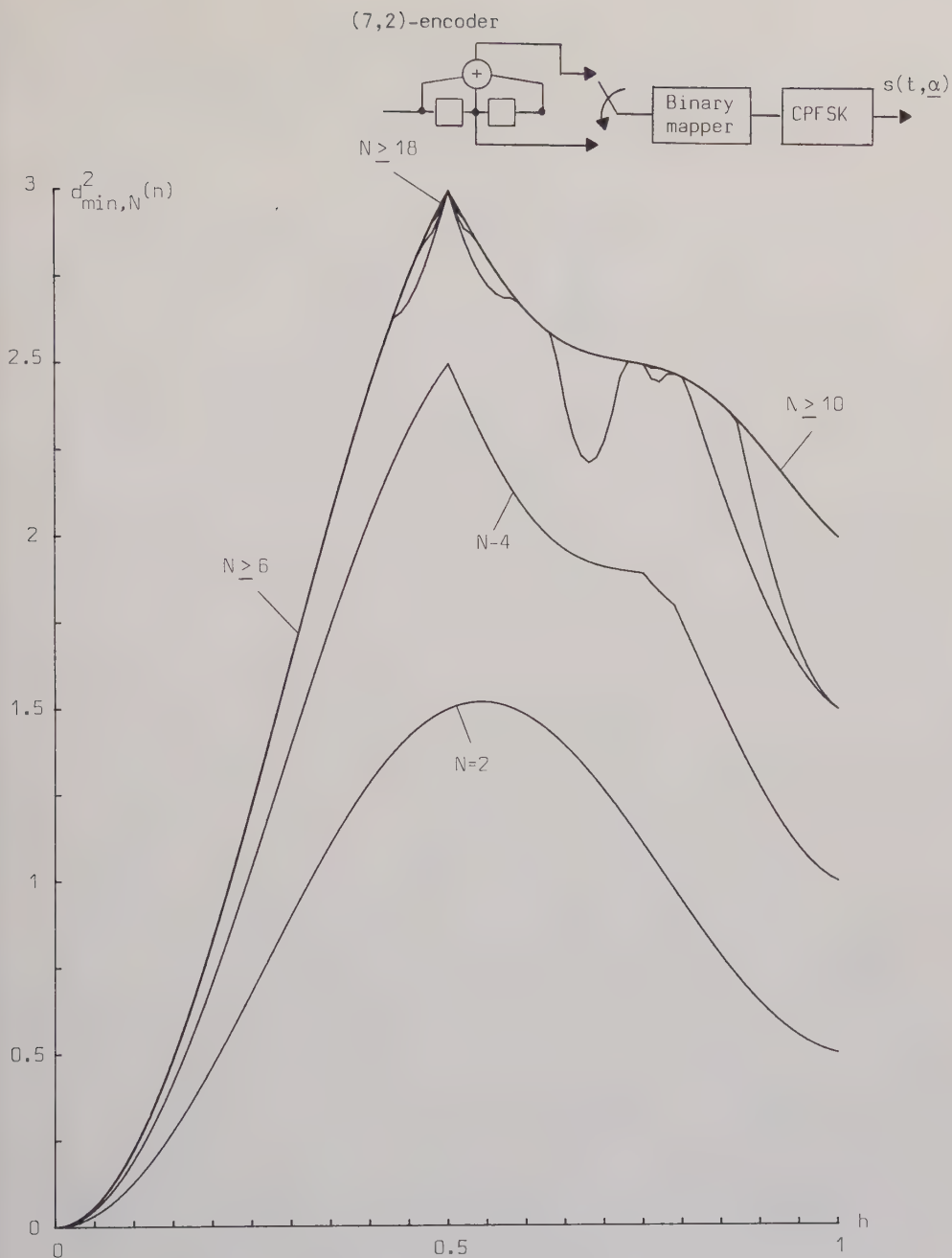


Figure 24. Minimum normalized squared Euclidean distance versus modulation index h when the (7,2)-encoder is used.

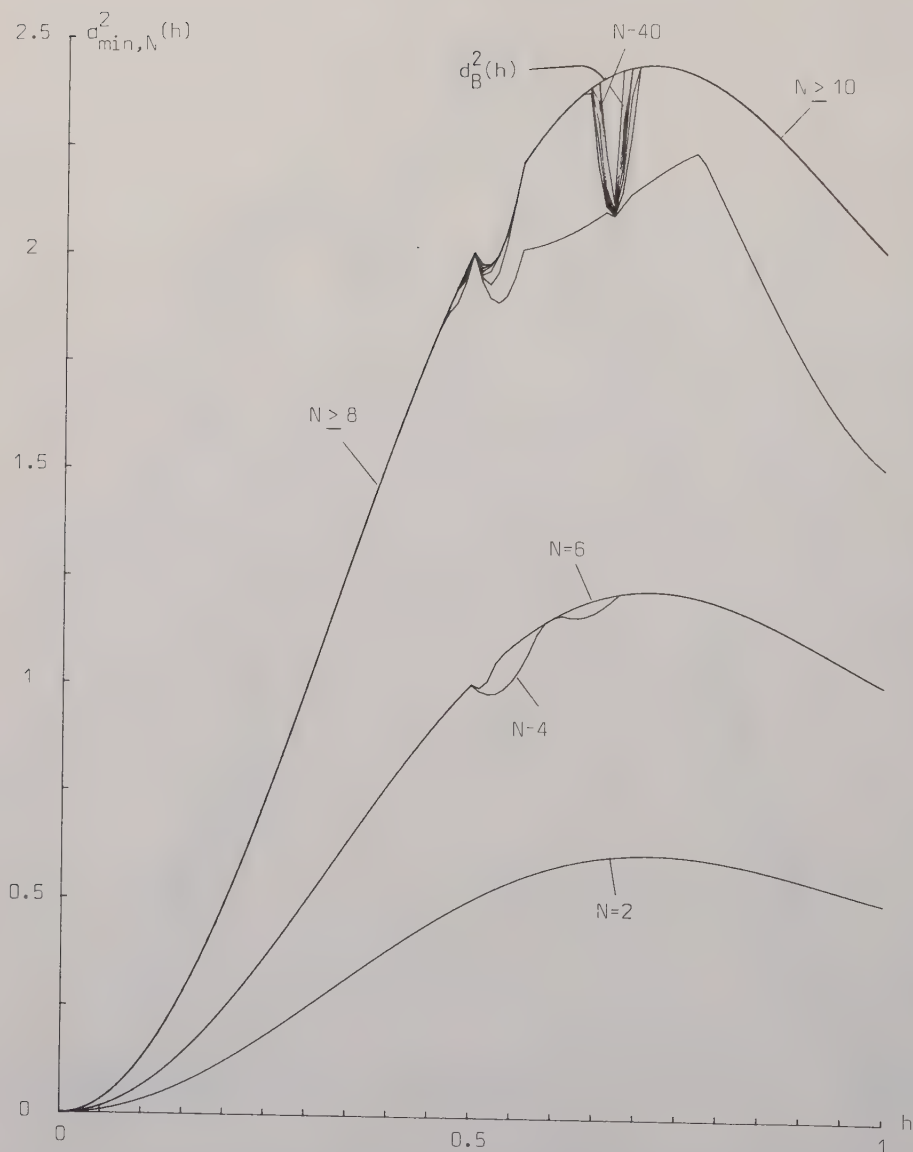
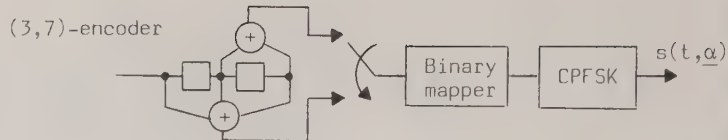


Figure 25. Minimum normalized squared Euclidean distance versus modulation index h when the (3,7)-encoder is used.

(7,6)-encoder

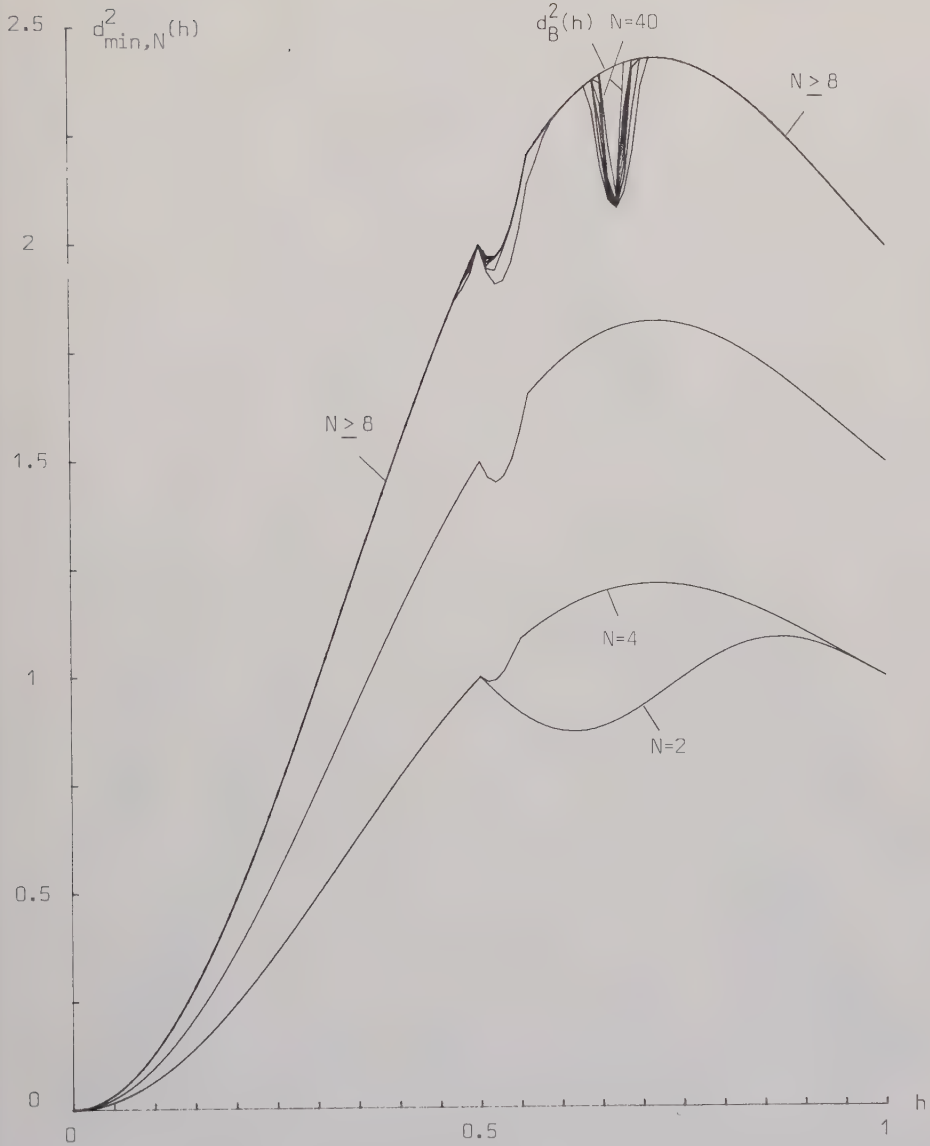
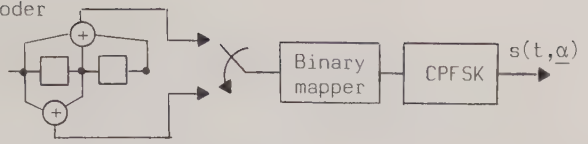


Figure 26. Minimum normalized squared Euclidean distance versus modulation index h when the (7,6)-encoder is used.

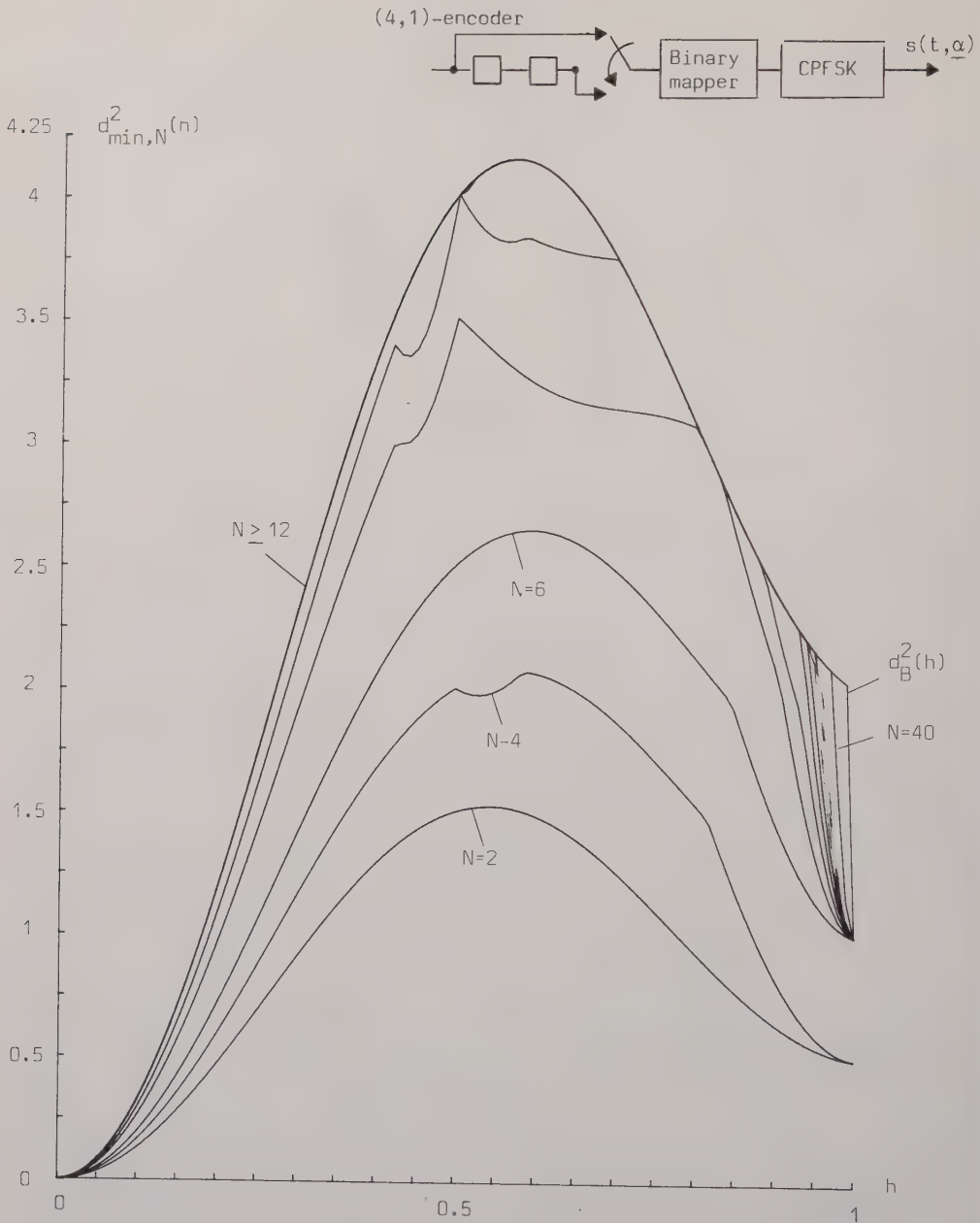


Figure 27. Minimum normalized squared Euclidean distance versus modulation index h when the (4,1)-encoder is used.

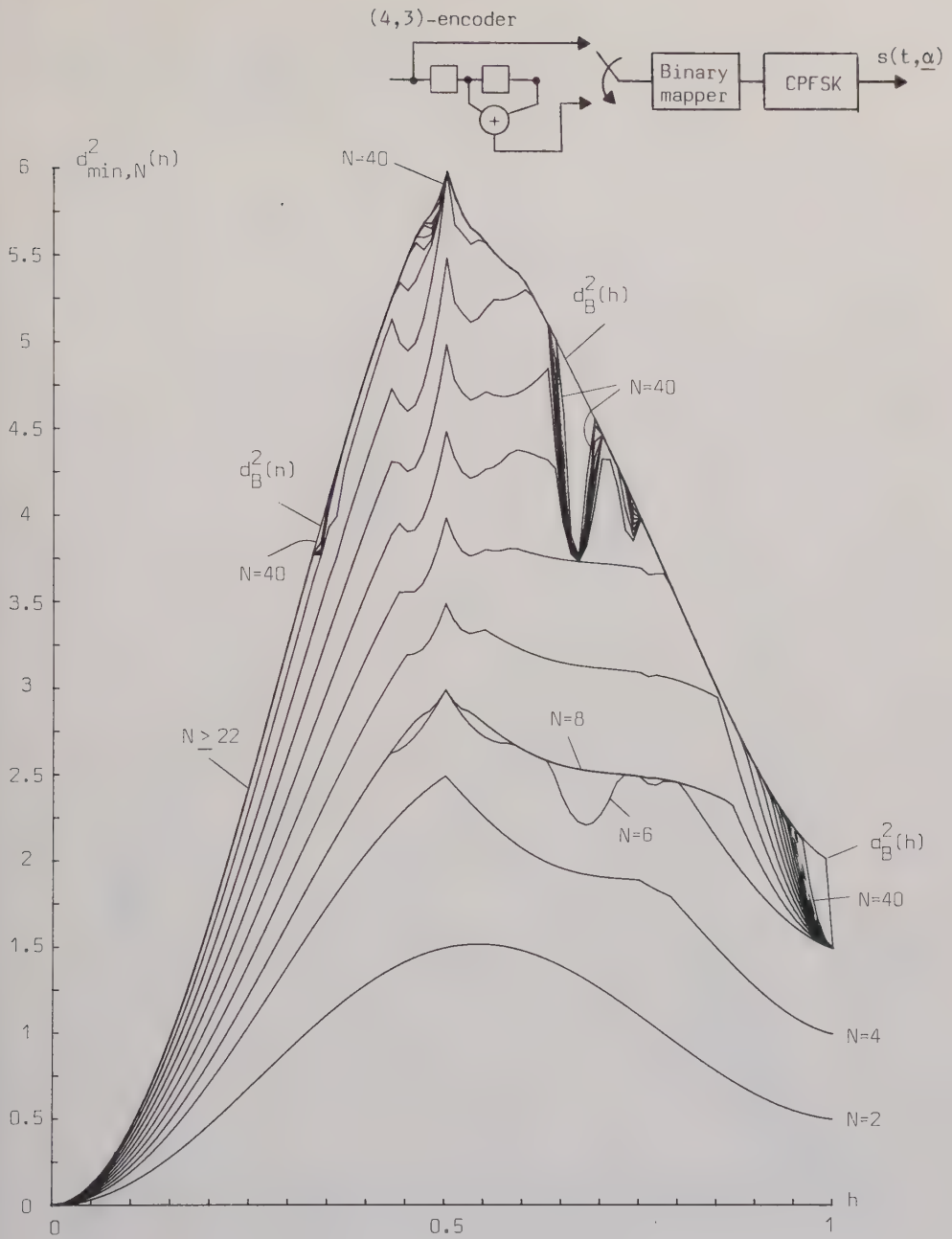


Figure 28. Minimum normalized squared Euclidean distance versus modulation index h when the (4,3)-encoder is used.

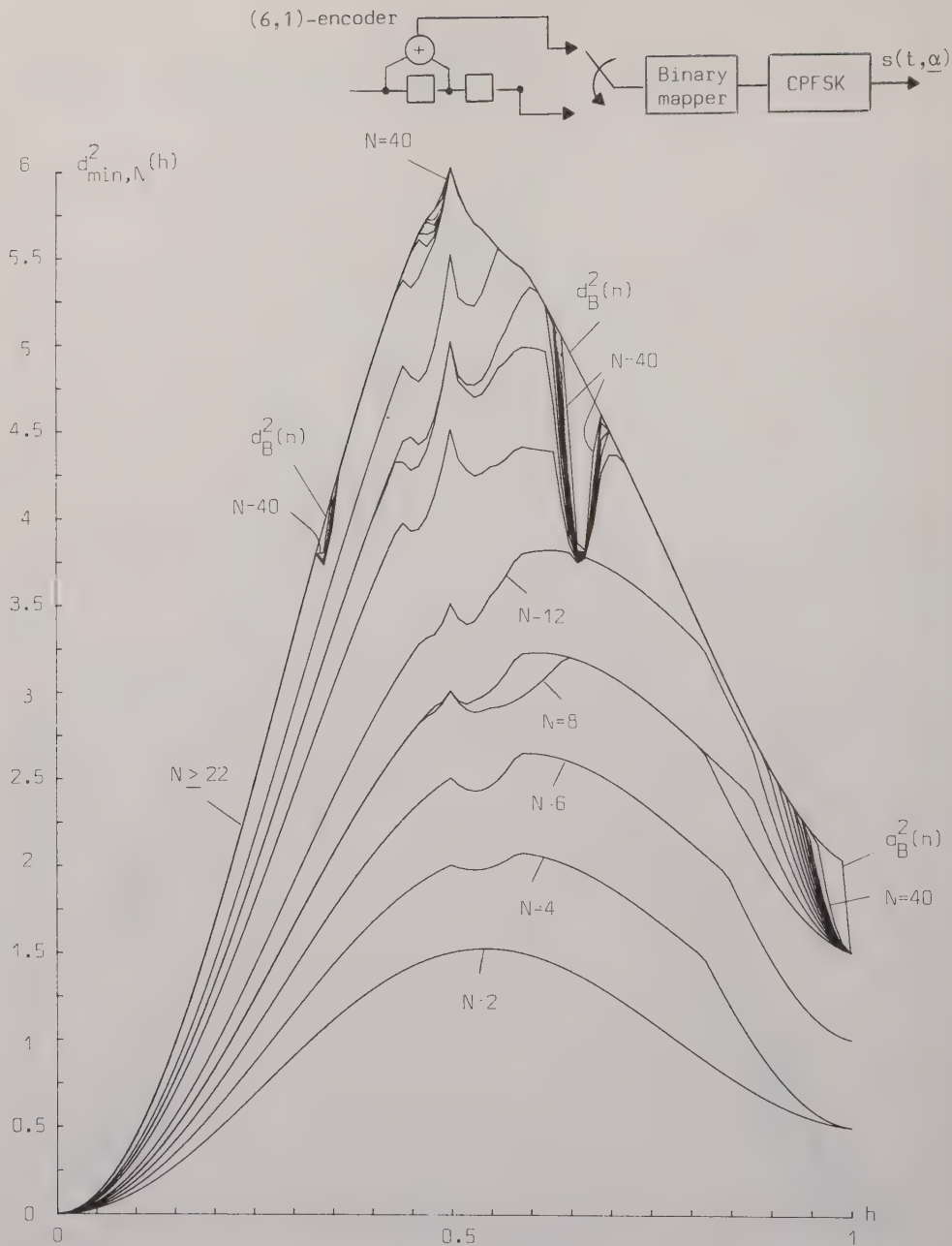


Figure 29. Minimum normalized squared Euclidean distance versus modulation index h when the (6,1)-encoder is used.

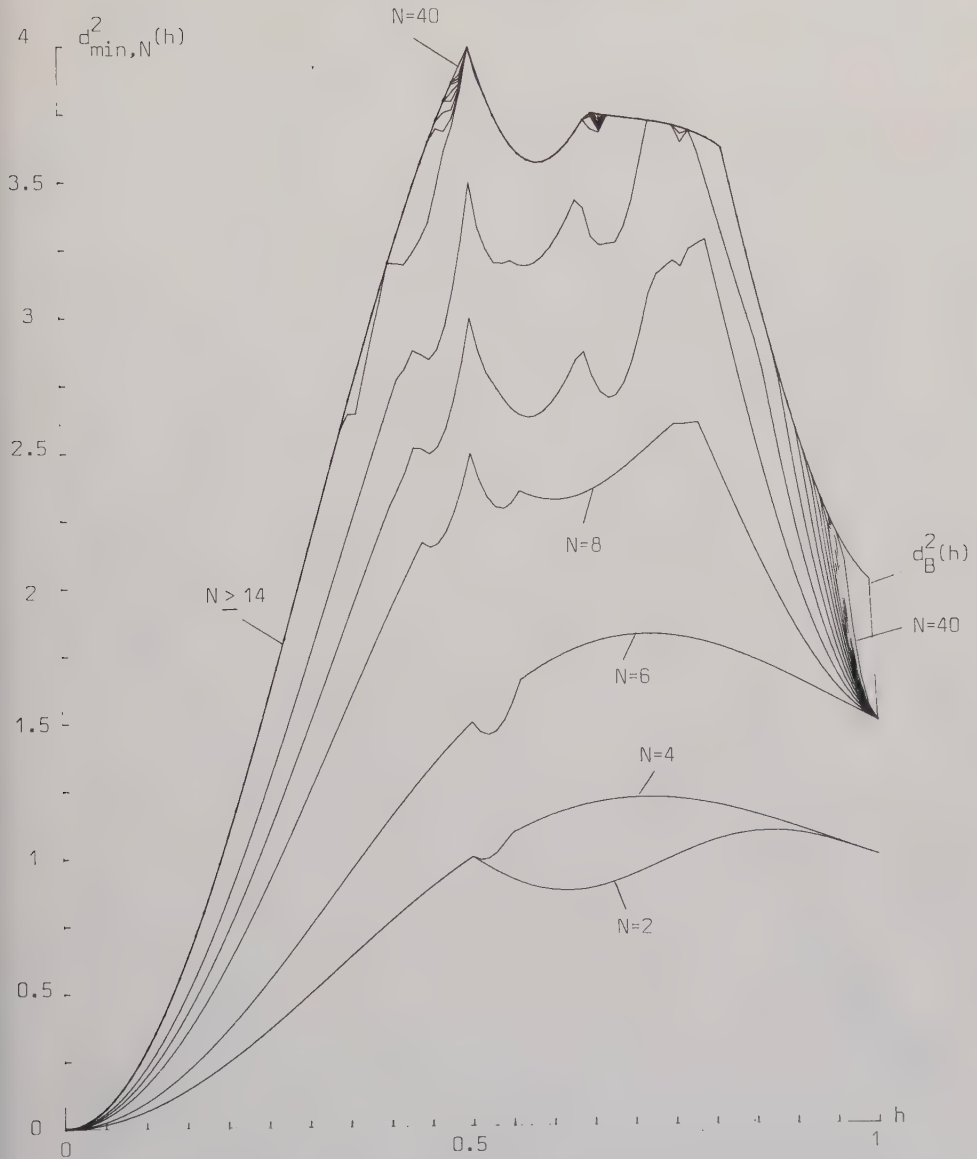
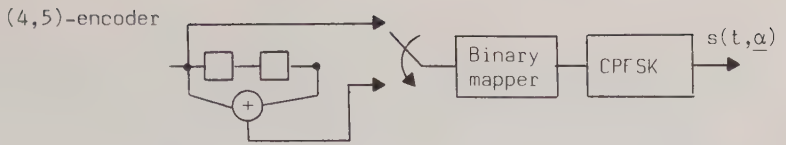


Figure 30. Minimum normalized squared Euclidean distance versus modulation index h when the (4,5)-encoder is used.

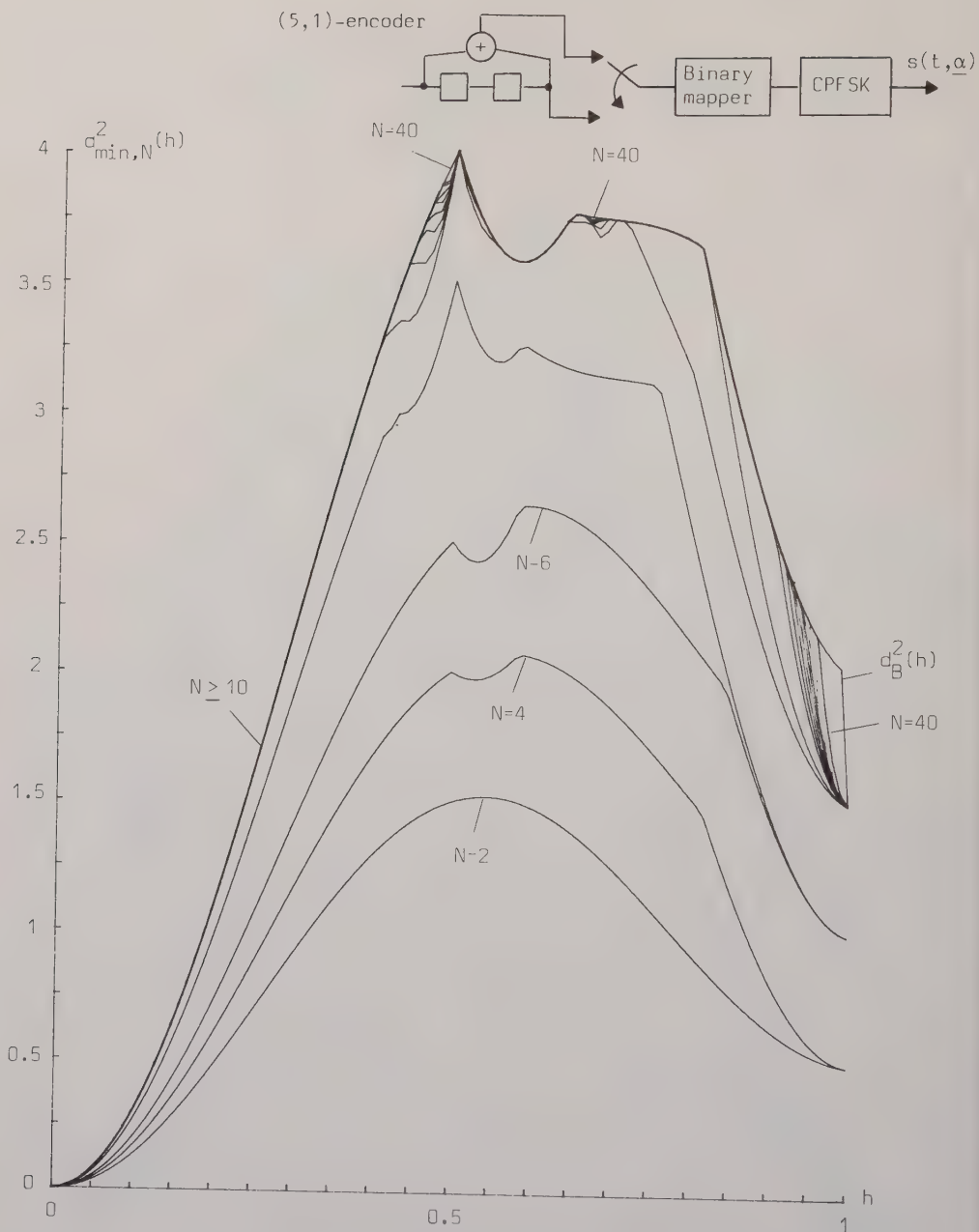


Figure 31. Minimum normalized squared Euclidean distance versus modulation index h when the (5,1)-encoder is used.

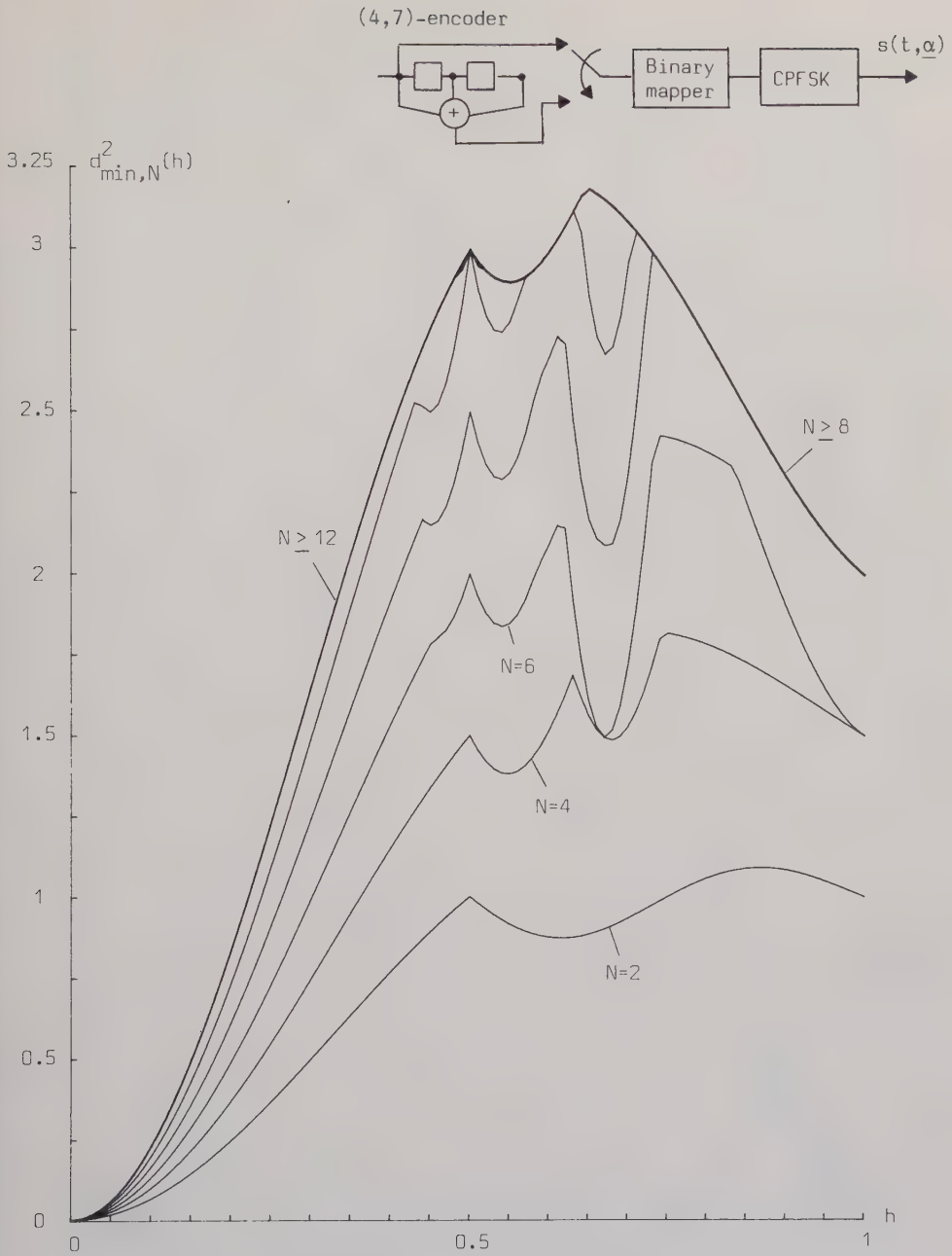


Figure 32. Minimum normalized squared Euclidean distance versus modulation index h when the (4,7)-encoder is used.

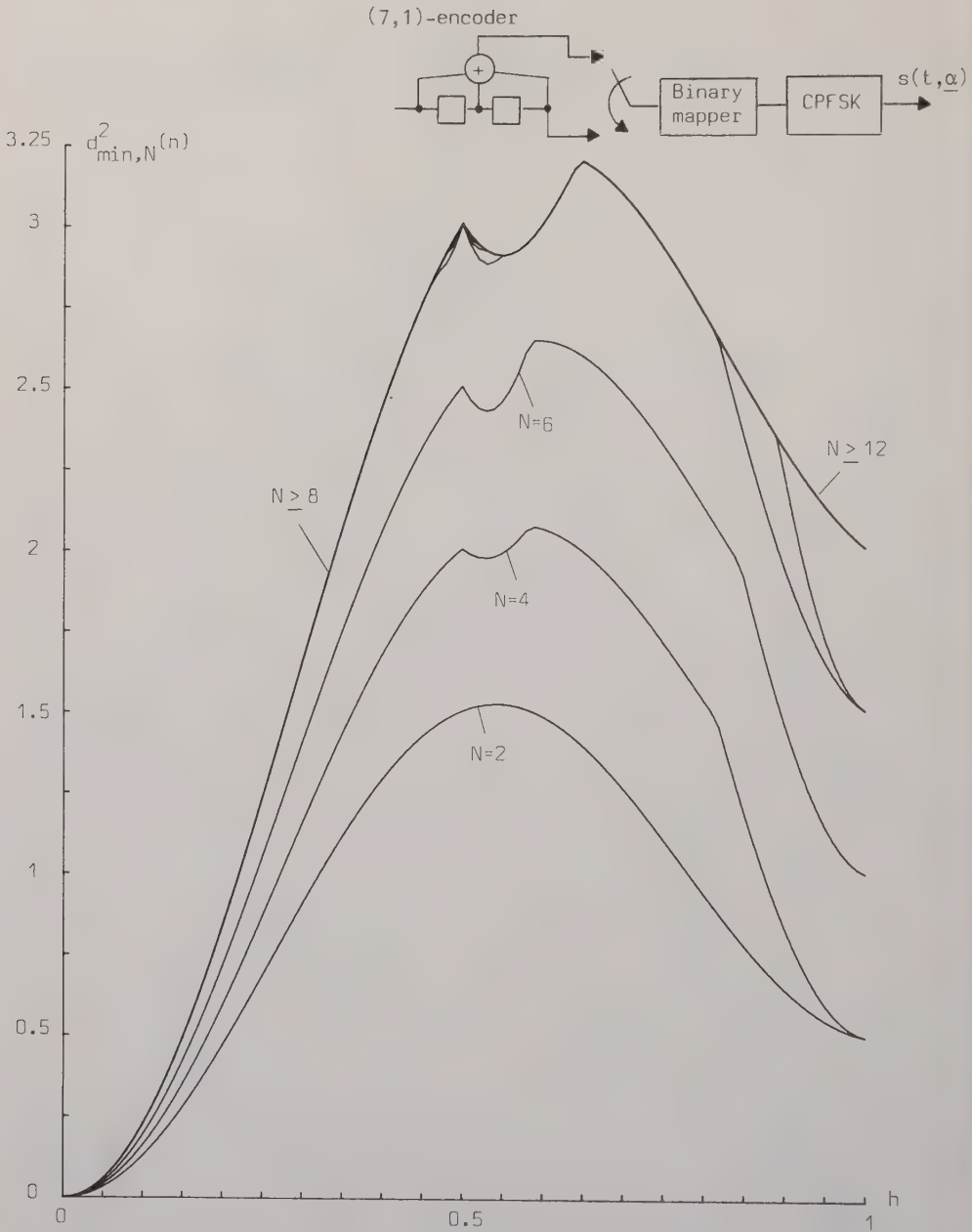


Figure 33. Minimum normalized squared Euclidean distance versus modulation index h when the (7,1)-encoder is used.

(7,5)-encoder

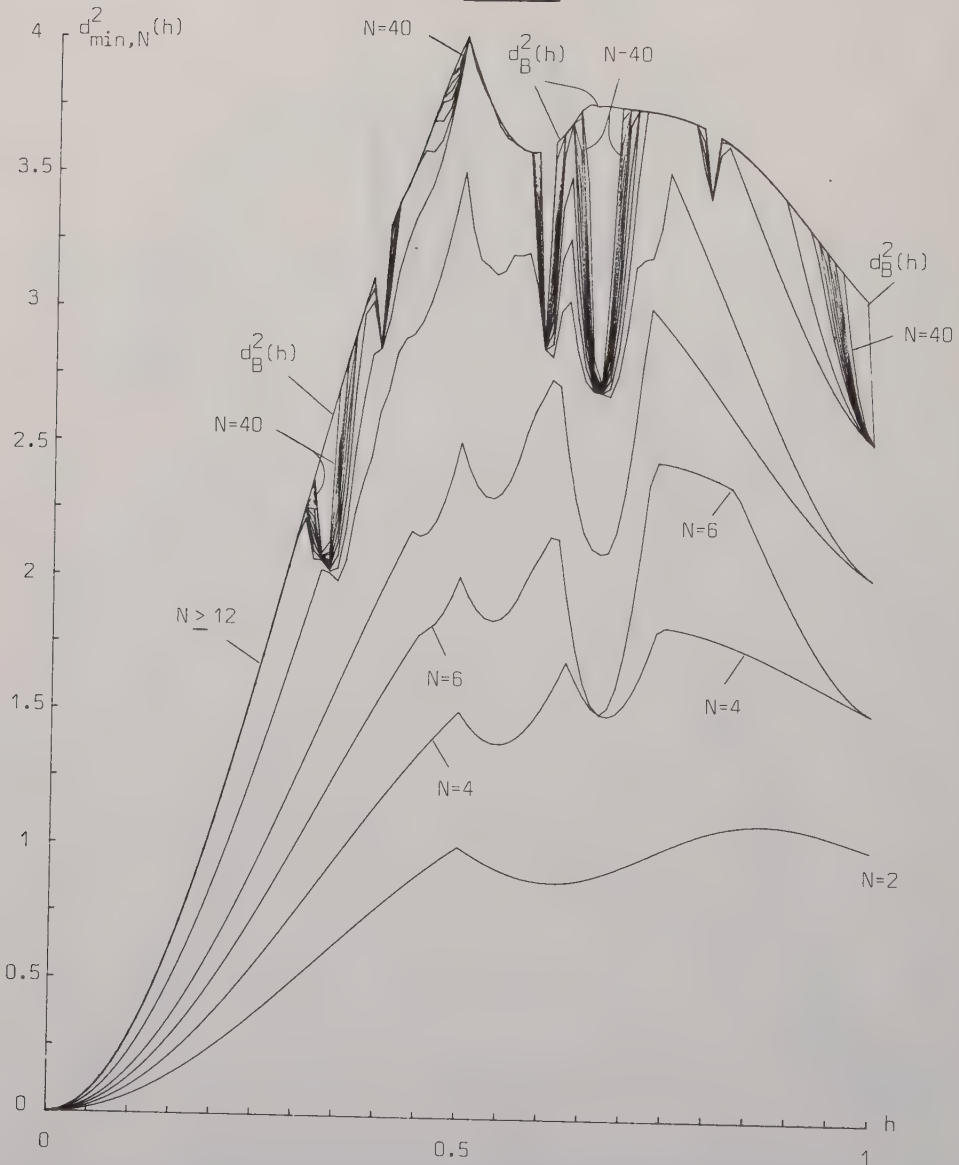
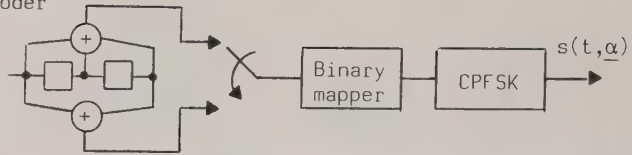


Figure 35. Minimum normalized squared Euclidean distance versus modulation index h when the (7,5)-encoder is used.

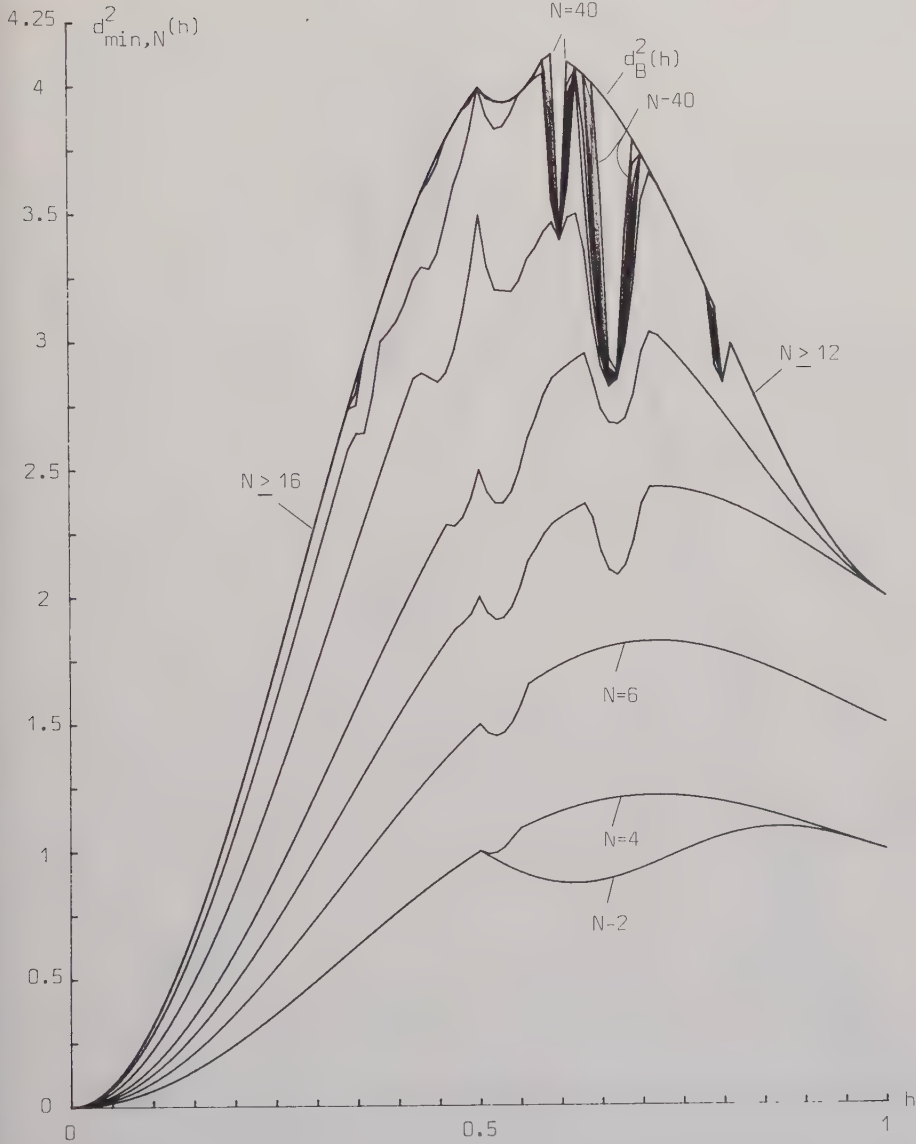
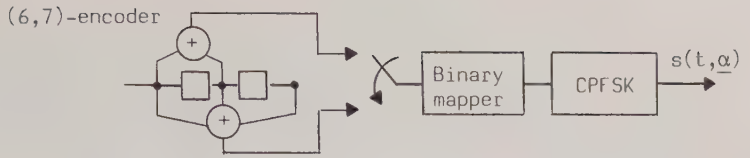


Figure 36. Minimum normalized squared Euclidean distance versus modulation index h when the (6,7)-encoder is used.

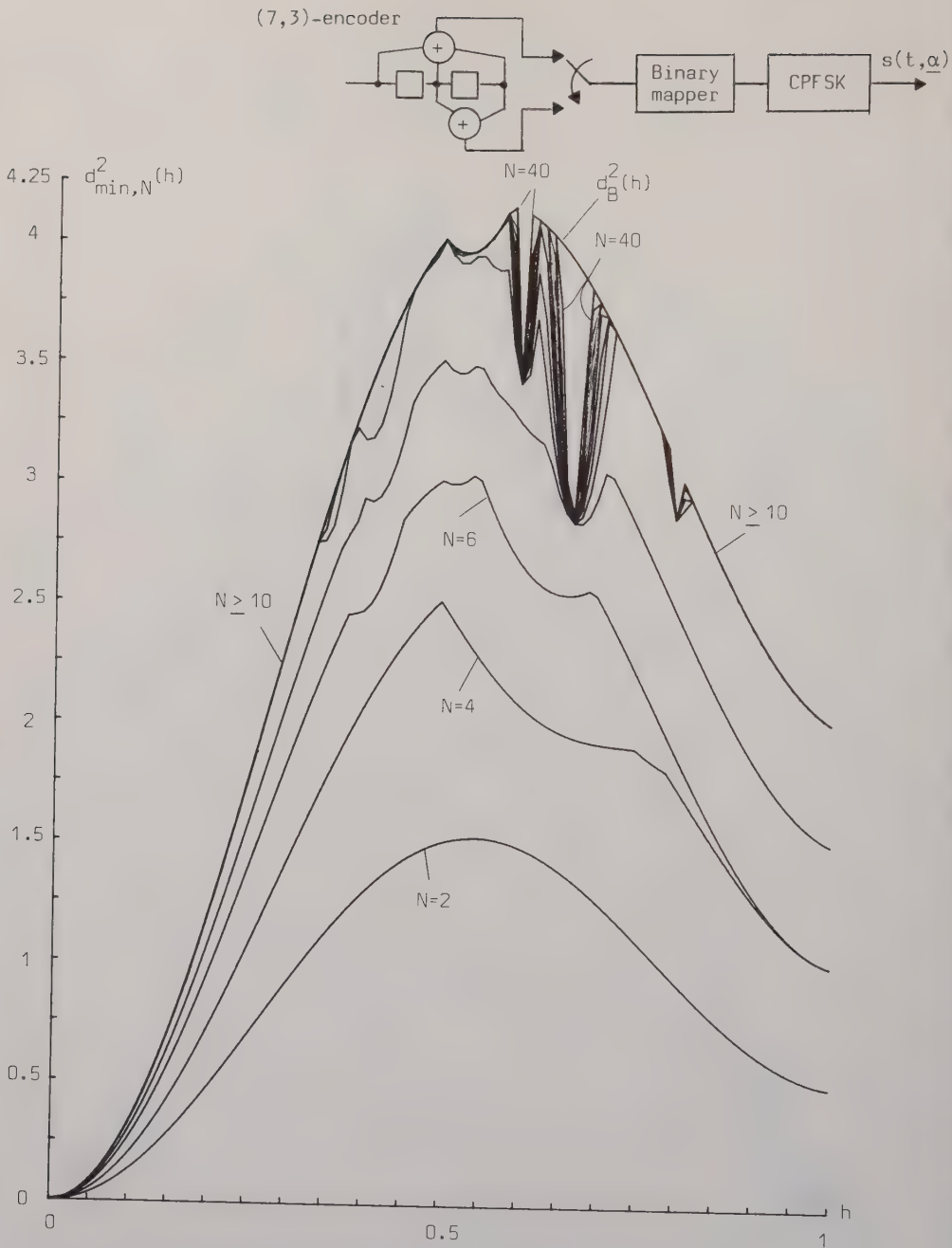


Figure 37. Minimum normalized squared Euclidean distance versus modulation index h when the (7,3)-encoder is used.

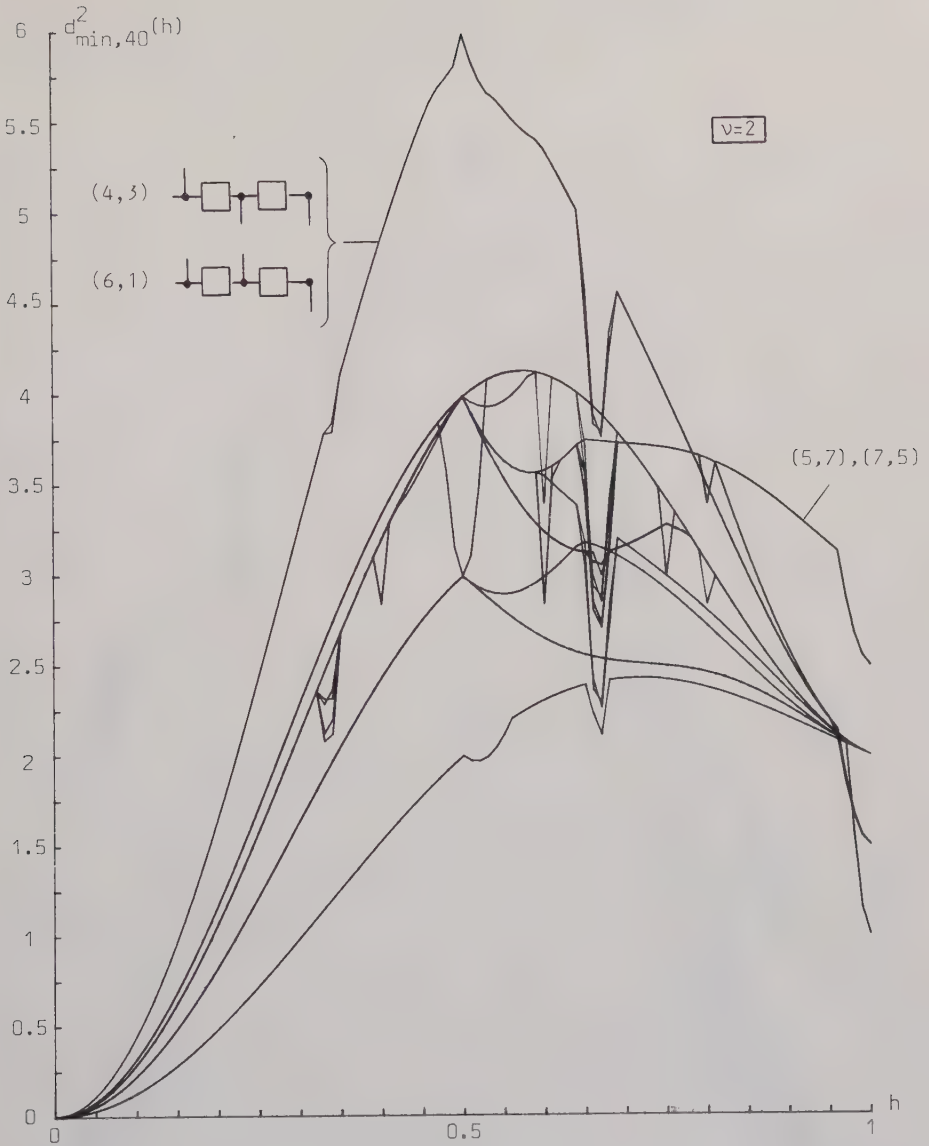


Figure 38. Minimum normalized squared Euclidean distance versus modulation index h for all noncatastrophic $v=2$ encoders, $N=40$.

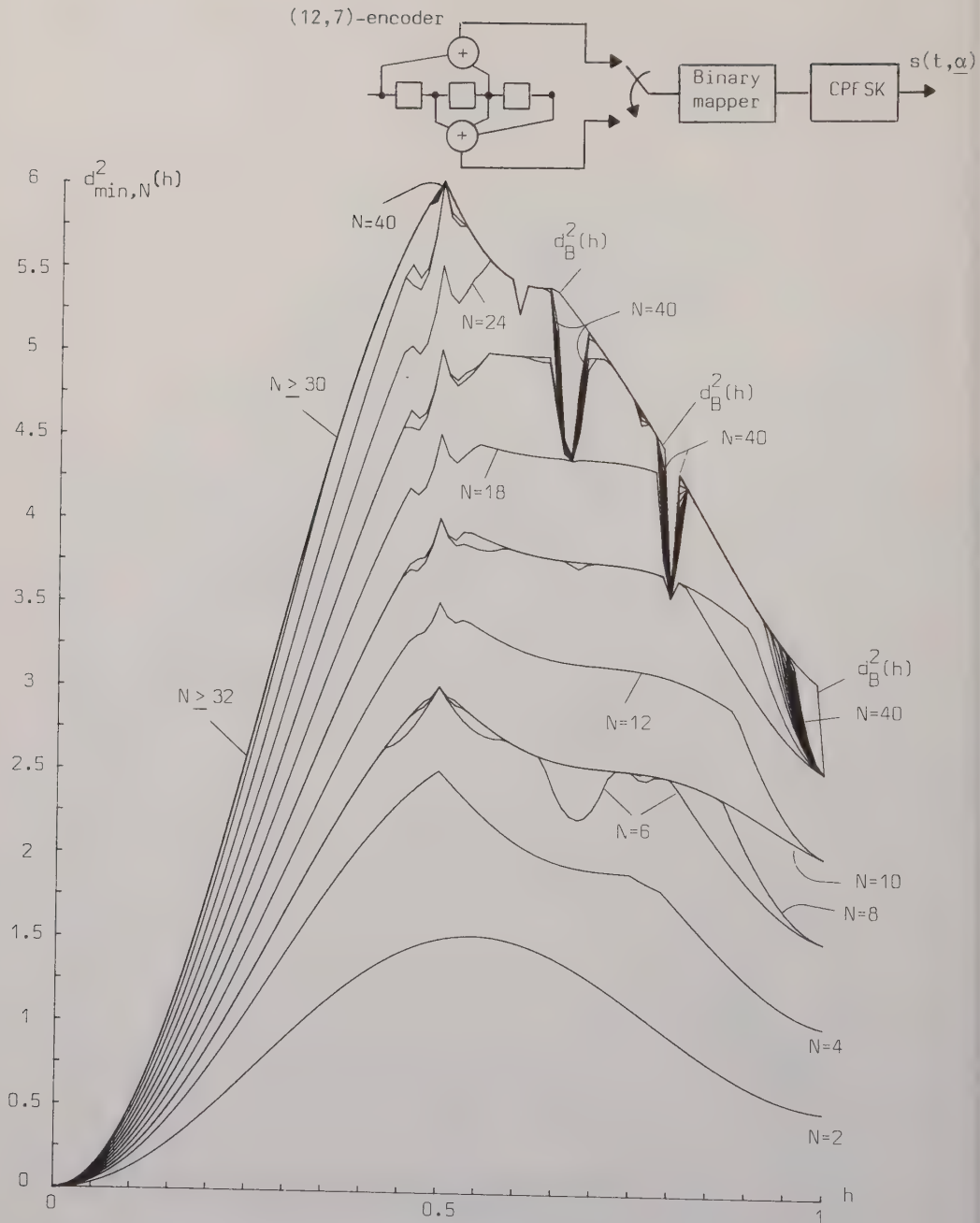
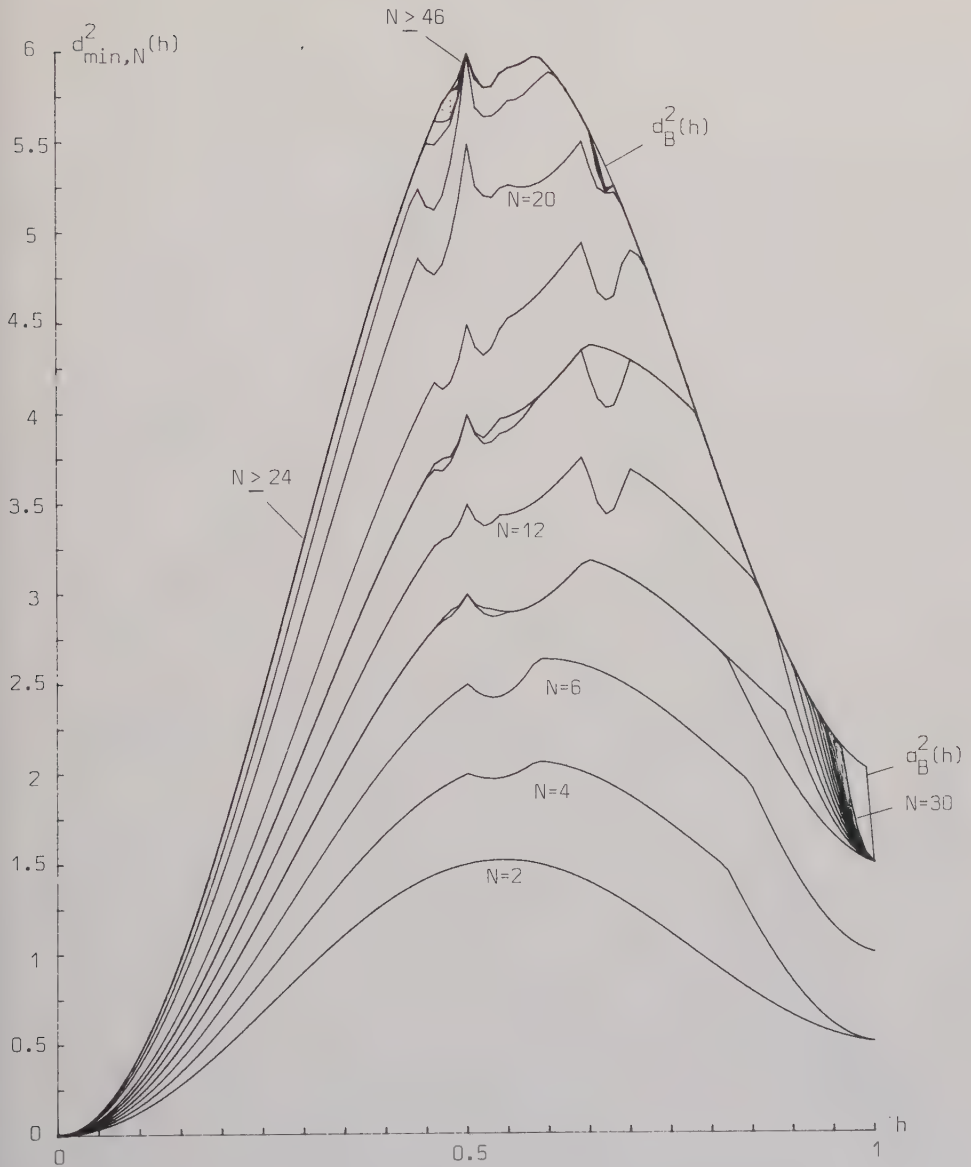
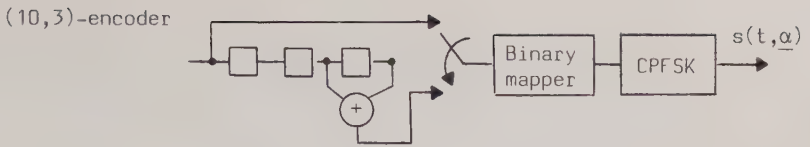


Figure 39. Minimum normalized squared Euclidean distance versus modulation index h when the (12,7)-encoder is used.



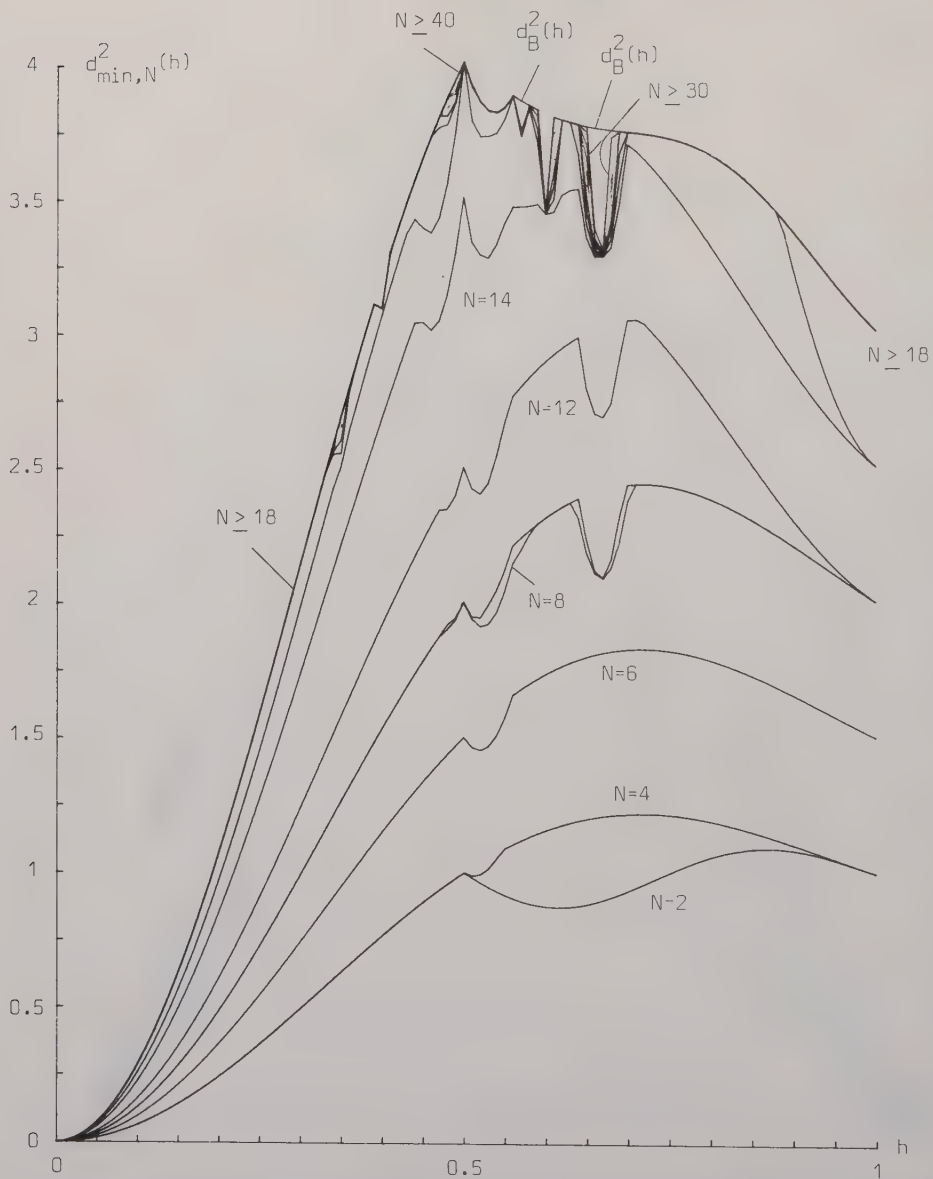
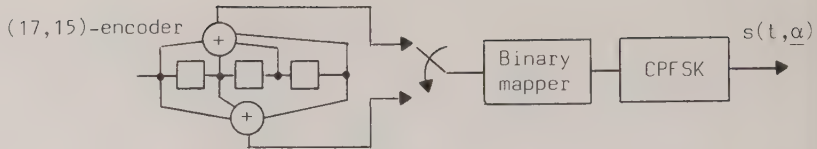


Figure 41. Minimum normalized squared Euclidean distance versus modulation index h when the (17,15)-encoder is used.

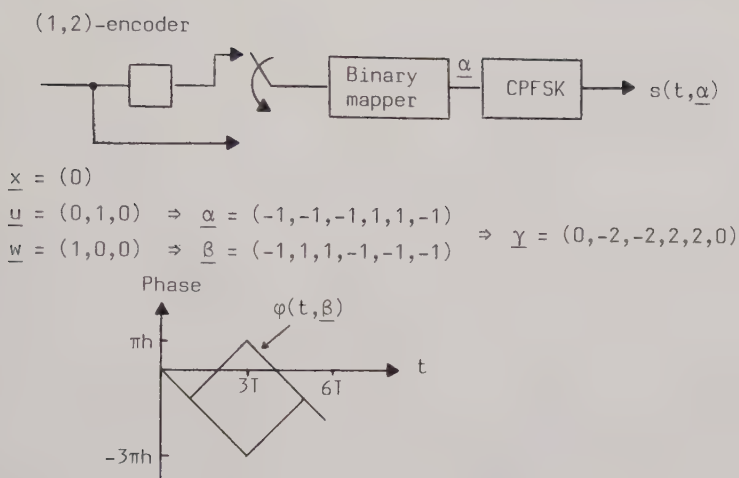
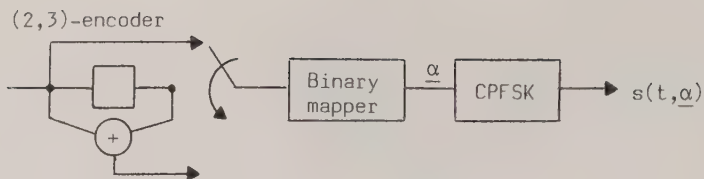


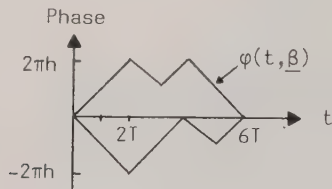
Figure 42. A minimum distance pair when the modulation index is small ($h \lesssim 0.30$).



$$\underline{x} = (0)$$

$$\underline{u} = (0, 1, 0) \Rightarrow \underline{\alpha} = (-1, -1, 1, 1, -1, 1)$$

$$\underline{w} = (1, 0, 0) \Rightarrow \underline{\beta} = (1, 1, -1, 1, -1, -1) \Rightarrow \underline{\gamma} = (-2, -2, 2, 0, 0, 2)$$



$$\underline{x} = (0)$$

$$\underline{u} = (0, 0, 1, 0) \Rightarrow \underline{\alpha} = (-1, -1, -1, -1, 1, 1, -1, 1)$$

$$\underline{w} = (1, 0, 0, 0) \Rightarrow \underline{\beta} = (1, 1, -1, 1, -1, -1, -1, -1) \Rightarrow \underline{\gamma} = (-2, -2, 0, -2, 2, 2, 0, 2)$$

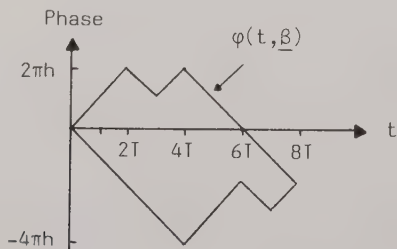
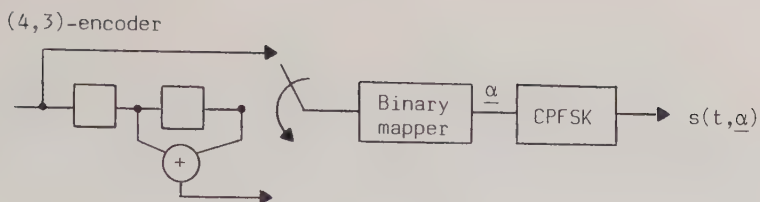


Figure 43. Two minimum distance pairs when the modulation index $h=1/2$.



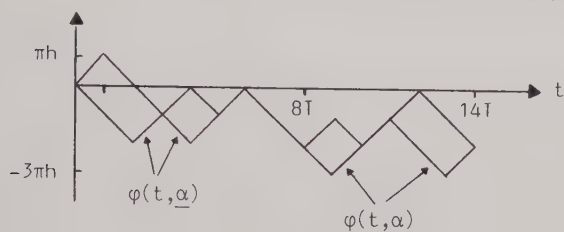
$$\underline{x} = (0,0)$$

$$\underline{u} = (0,1,1,0,0,1,0) \Rightarrow \underline{\alpha} = (-1,-1,1,-1,1,1,-1,-1,-1,1,1,-1,-1,1)$$

$$\underline{w} = (1,0,0,0,1,1,0) \Rightarrow \underline{\beta} = (1,-1,-1,1,-1,1,-1,-1,1,-1,1,1,-1,-1)$$

Phase

$$\underline{\gamma} = (-2,0,2,-2,2,0,0,0,-2,2,0,-2,0,2)$$



$$\underline{x} = (0,0)$$

$$\underline{u} = (0,1,1,0,0,0,1,0) \Rightarrow \underline{\alpha} = (-1,-1,1,-1,1,1,-1,-1,-1,1,-1,-1,1,-1,-1)$$

$$\underline{w} = (1,0,0,0,0,1,1,0) \Rightarrow \underline{\beta} = (1,-1,-1,1,-1,1,-1,-1,-1,1,-1,1,1,-1,-1)$$

Phase

$$\underline{\gamma} = (-2,0,2,-2,2,0,0,0,0,2,-2,0,0,-2,0,2)$$

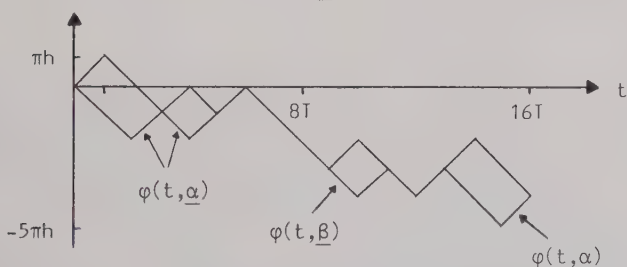
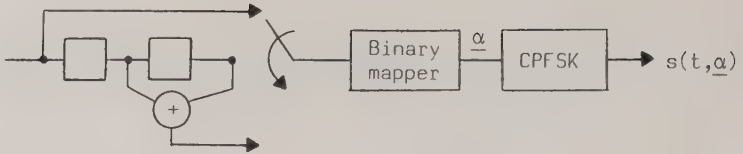


Figure 44. Two minimum distance pairs when $h=0.20$.

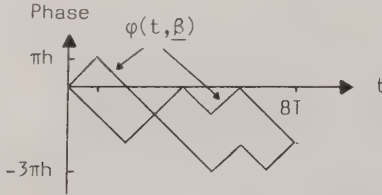
(4,3)-encoder



$$\underline{x} = (0,0)$$

$$\underline{u} = (0,1,0,0) \Rightarrow \underline{\alpha} = (-1,-1,1,-1,-1,1,-1,1) \Rightarrow \underline{\gamma} = (-2,0,2,-2,0,0,0,2)$$

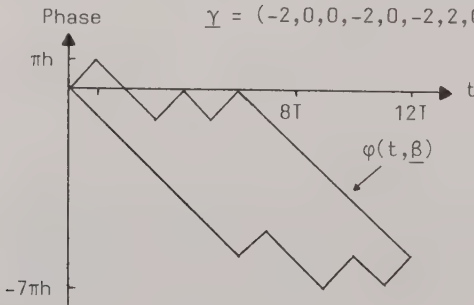
$$\underline{w} = (1,0,0,0) \Rightarrow \underline{\beta} = (1,-1,-1,1,-1,1,-1,-1)$$



$$\underline{x} = (0,0)$$

$$\underline{u} = (0,0,0,1,0,0) \Rightarrow \underline{\alpha} = (-1,-1,-1,-1,-1,1,-1,-1,1,-1,1,-1,1)$$

$$\underline{w} = (1,0,0,0,0,0) \Rightarrow \underline{\beta} = (1,-1,-1,1,-1,1,-1,-1,-1,-1,-1,-1,-1)$$



$$\underline{x} = (0,0)$$

$$\underline{u} = (0,0,0,1,0,1,0,0) \Rightarrow \underline{\alpha} = (-1,-1,-1,-1,-1,1,-1,-1,1,1,-1,1,-1,1)$$

$$\underline{w} = (1,1,1,1,0,0,0,0) \Rightarrow \underline{\beta} = (1,-1,1,1,1,-1,1,-1,-1,-1,-1,-1,-1,-1)$$

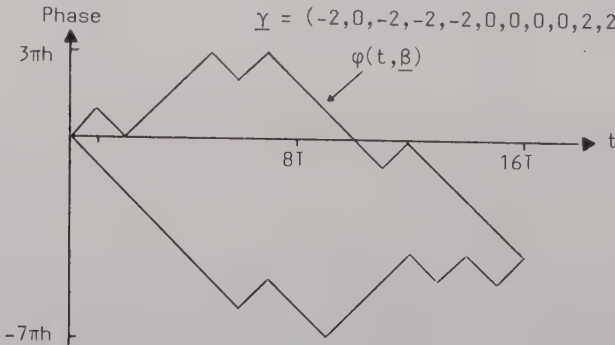
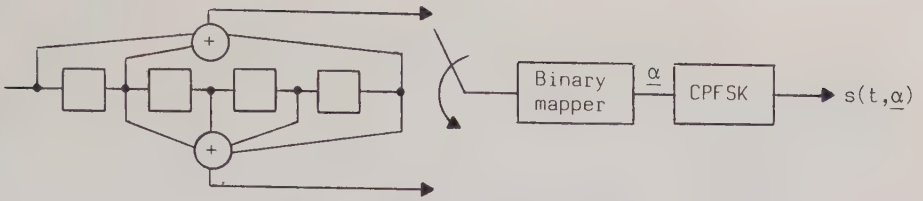


Figure 45. Three minimum distance pairs when $h=1/2$.

(31,17)-encoder

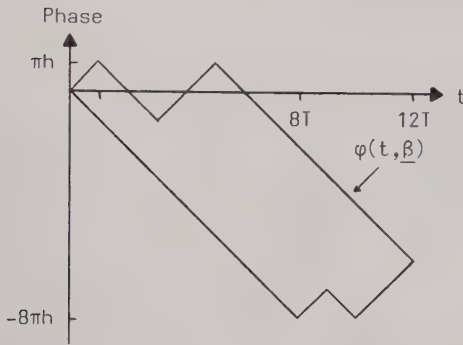


$$\underline{x} = (0,0,0,0)$$

$$\underline{u} = (0,0,0,0,1,0) \Rightarrow \underline{\alpha} = (-1,-1,-1,-1,-1,-1,-1,-1,1,-1,1,1)$$

$$\underline{w} = (1,1,0,0,1,0) \Rightarrow \underline{\beta} = (1,-1,-1,1,1,-1,-1,-1,-1,-1,-1,-1)$$

$$\underline{\gamma} = (-2,0,0,-2,-2,0,0,0,2,0,2,2)$$

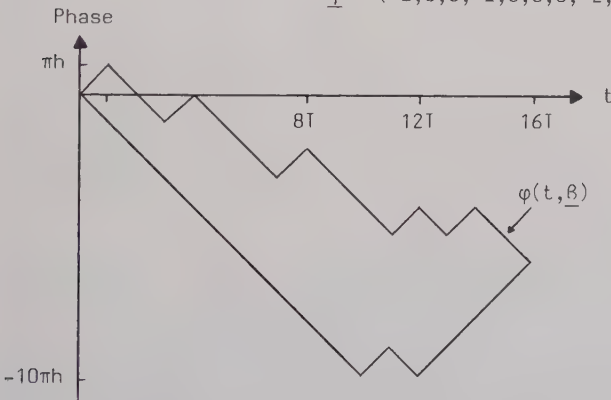


$$\underline{x} = (0,0,0,0)$$

$$\underline{u} = (0,0,0,0,0,1,0,1) \Rightarrow \underline{\alpha} = (-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,1,-1,1,1,1)$$

$$\underline{w} = (1,1,1,1,0,1,0,1) \Rightarrow \underline{\beta} = (1,-1,-1,1,-1,-1,-1,1,-1,-1,-1,1,-1,1,-1)$$

$$\underline{\gamma} = (-2,0,0,-2,0,0,0,-2,0,0,2,-2,2,0,2,2)$$

Figure 46. Two minimum distance pairs when $h=1/2$.

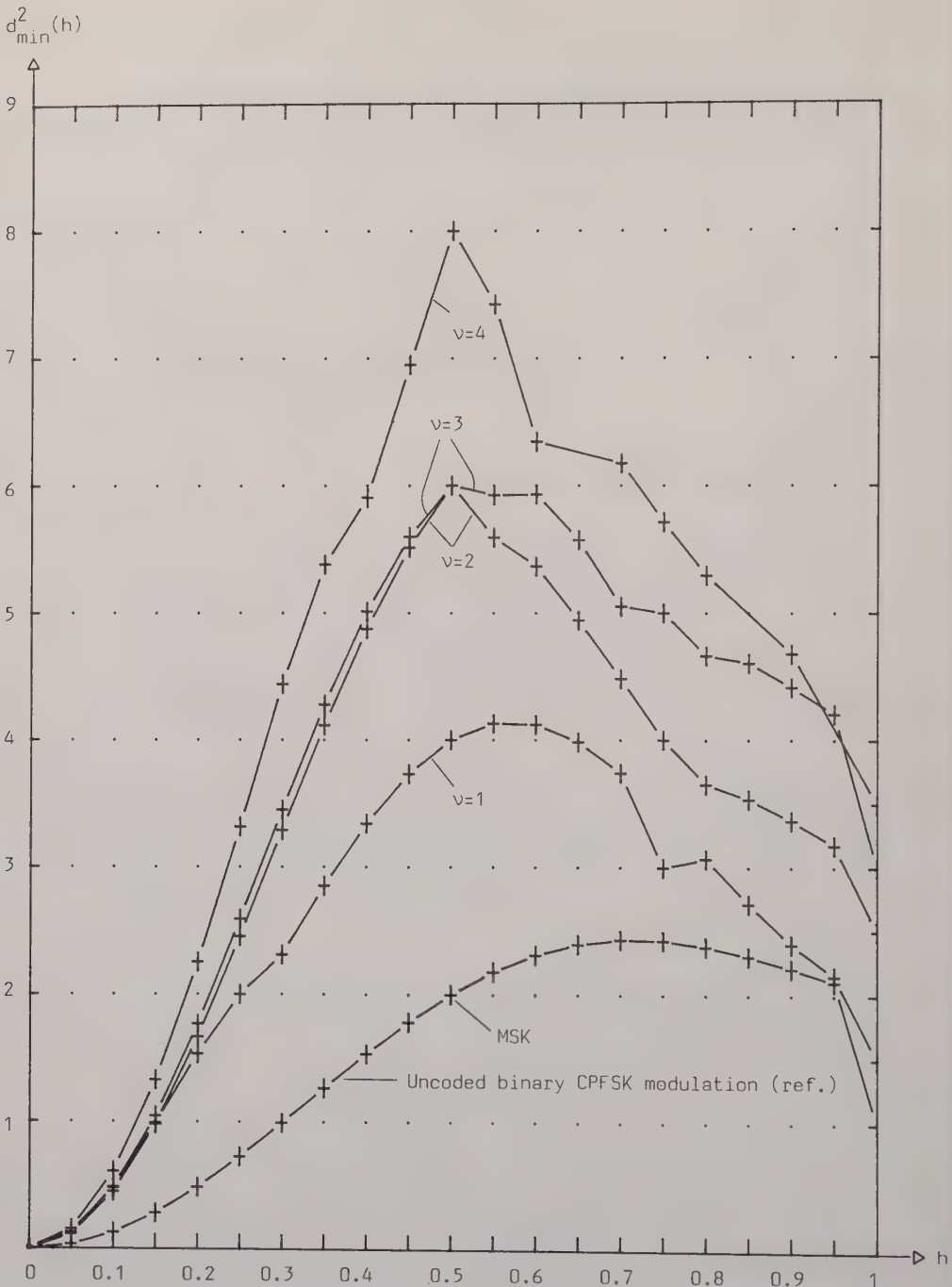


Figure 47. Rate 1/2 convolutional encoder and binary CPFSK modulation. The optimum $d_{\min}^2(h)$ for different v and h . See also tables 2-4. Note that the connection lines do not contain any numerical results.

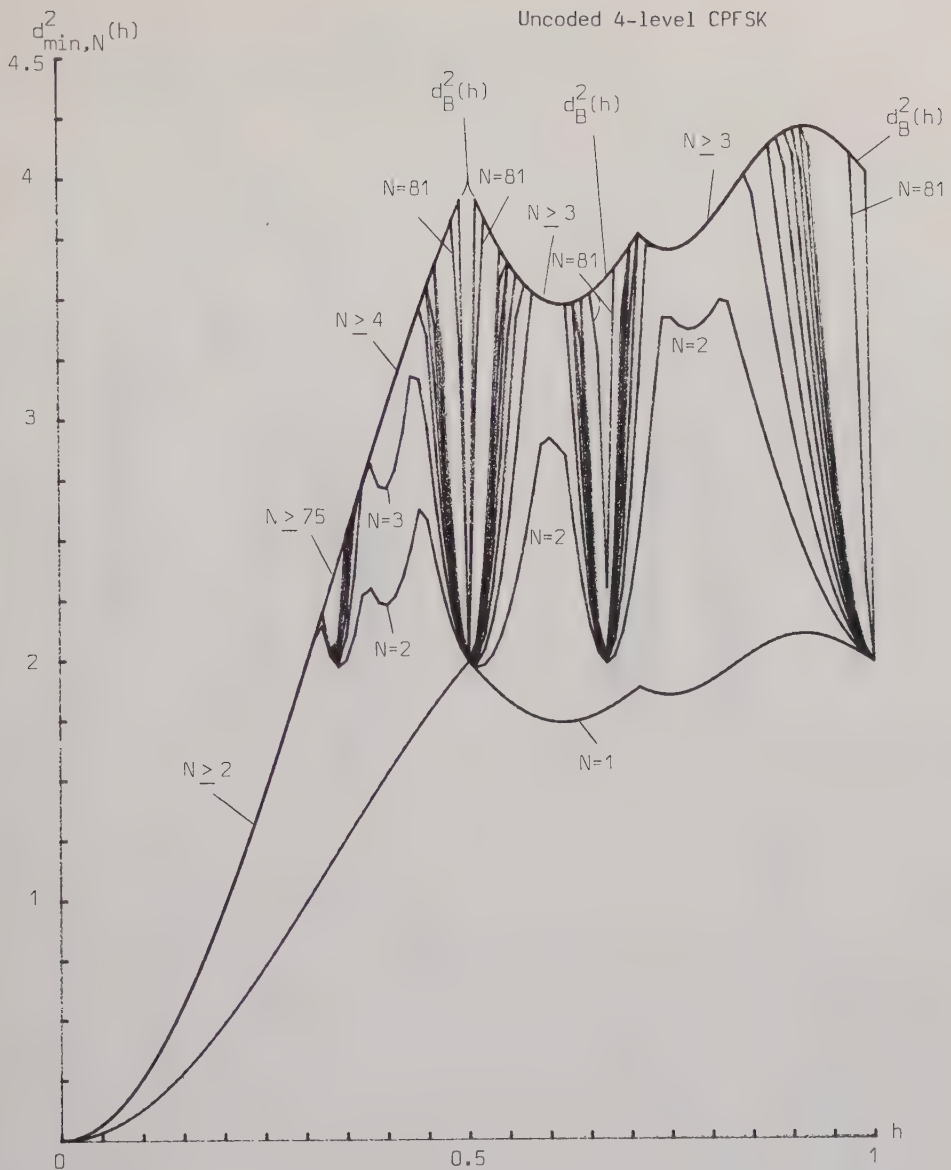


Figure 48. The minimum normalized squared Euclidean distance as a function of h , with N as parameter, for the uncoded 4-level CPFSK scheme.

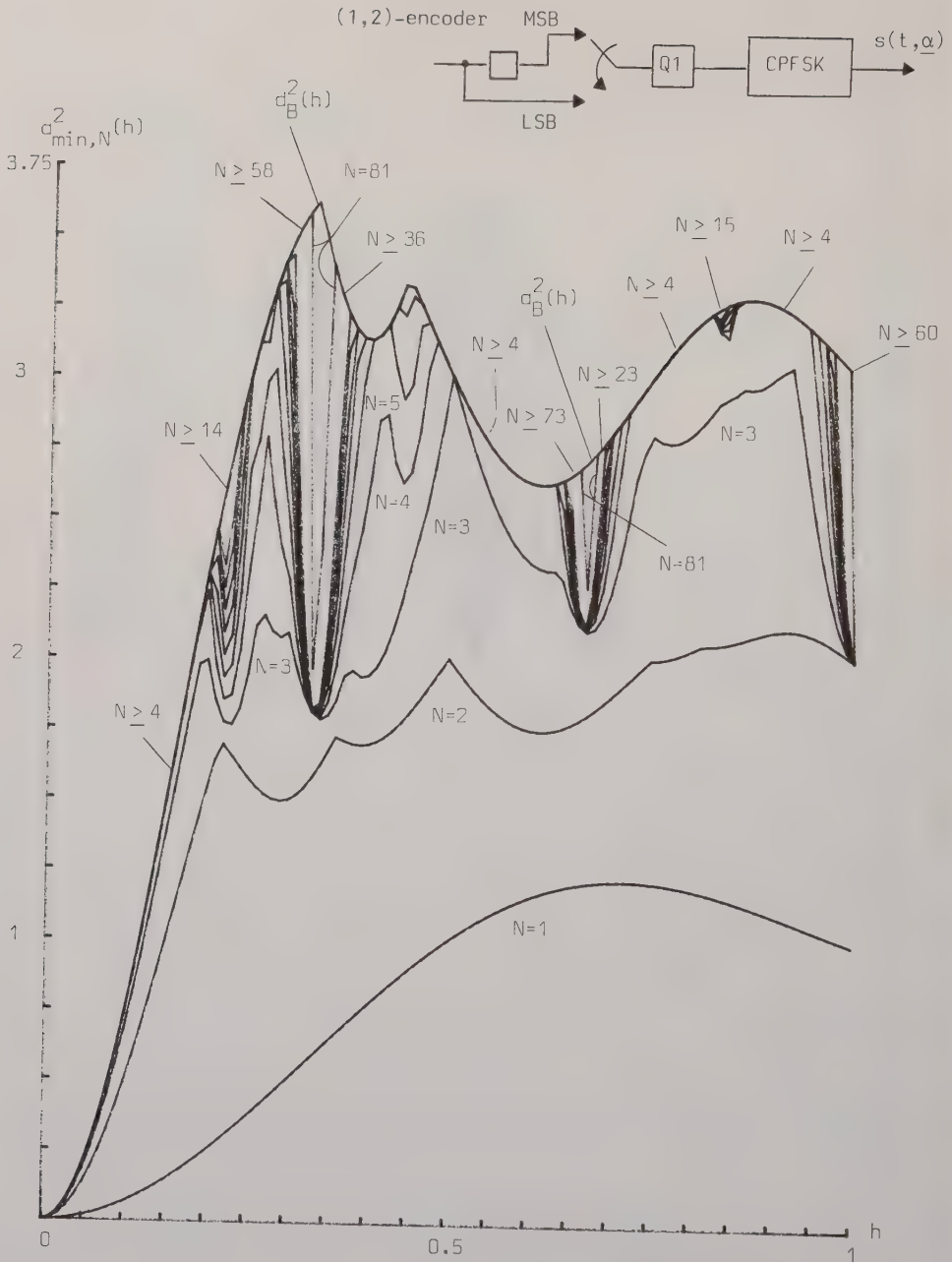


Figure 49. The minimum normalized squared Euclidean distance as a function of h , with N as parameter, for the combination $\{(1,2), Q1\}$ and CPFSK modulation.

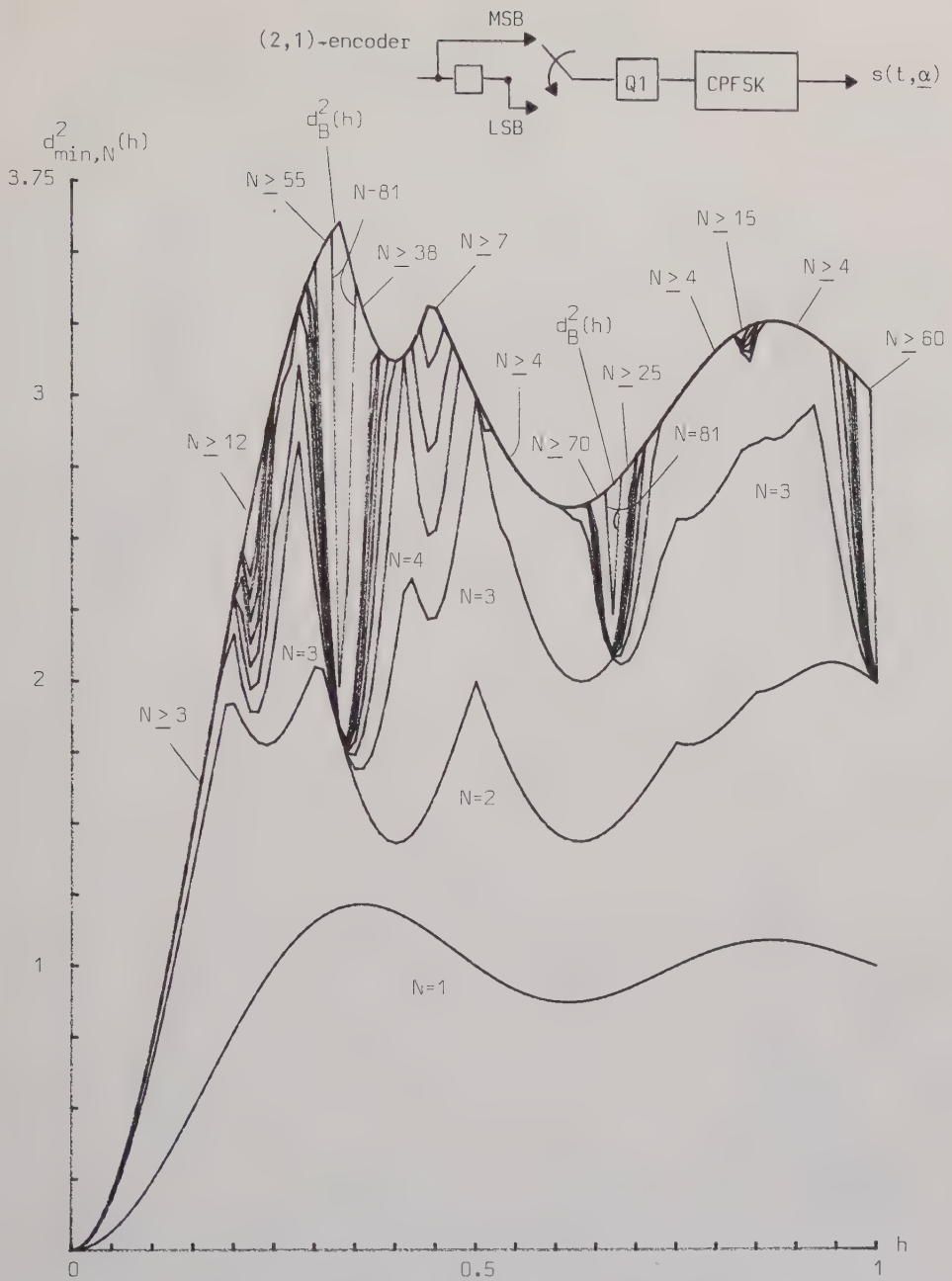


Figure 50. The minimum normalized squared Euclidean distance as a function of h , with N as parameter, for the combination $\{(2,1), Q1\}$ and CPFSK modulation.

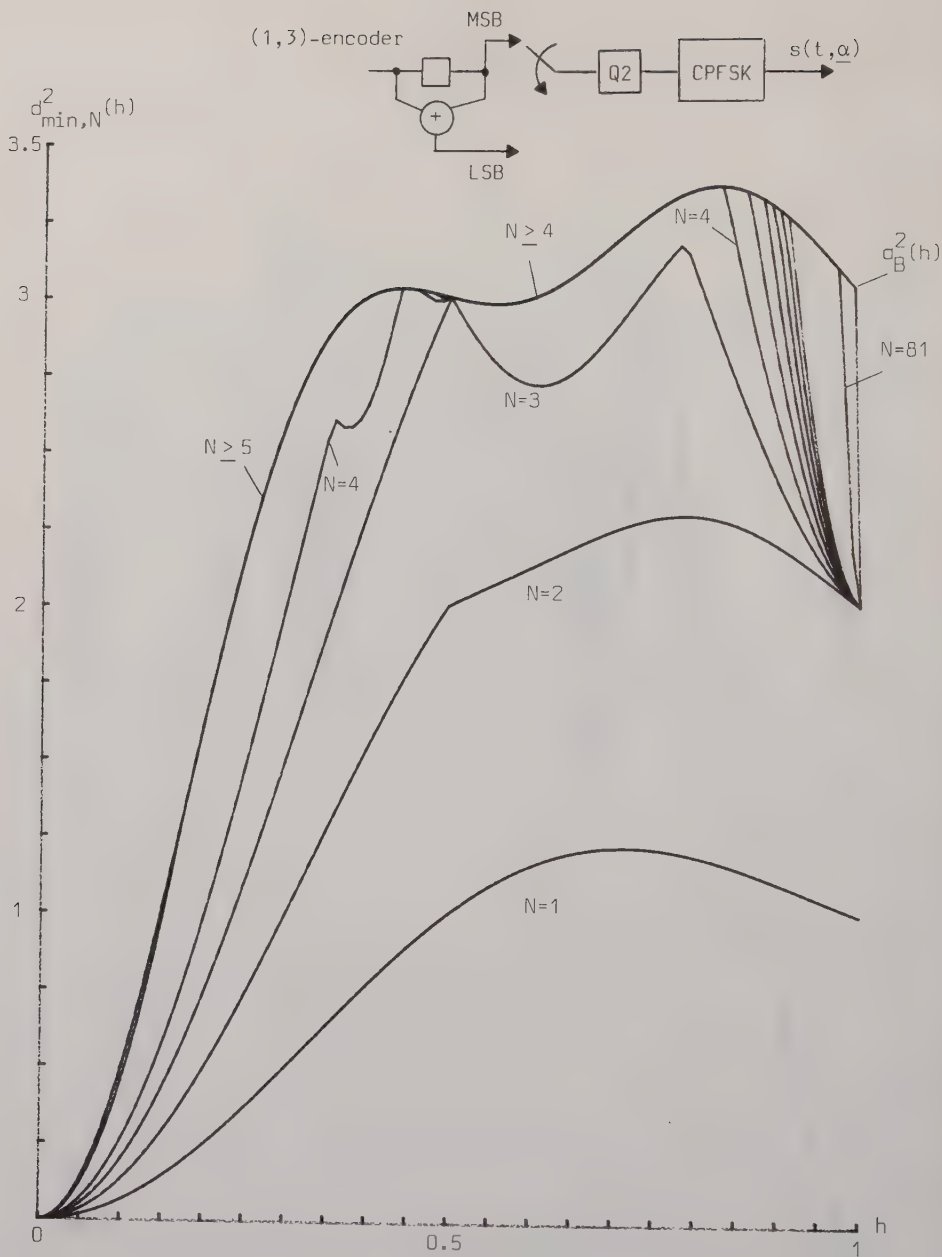


Figure 51. The minimum normalized squared Euclidean distance as a function of h , with N as parameter, for the combination $\{(1,3), Q2\}$ and CPFSK modulation.

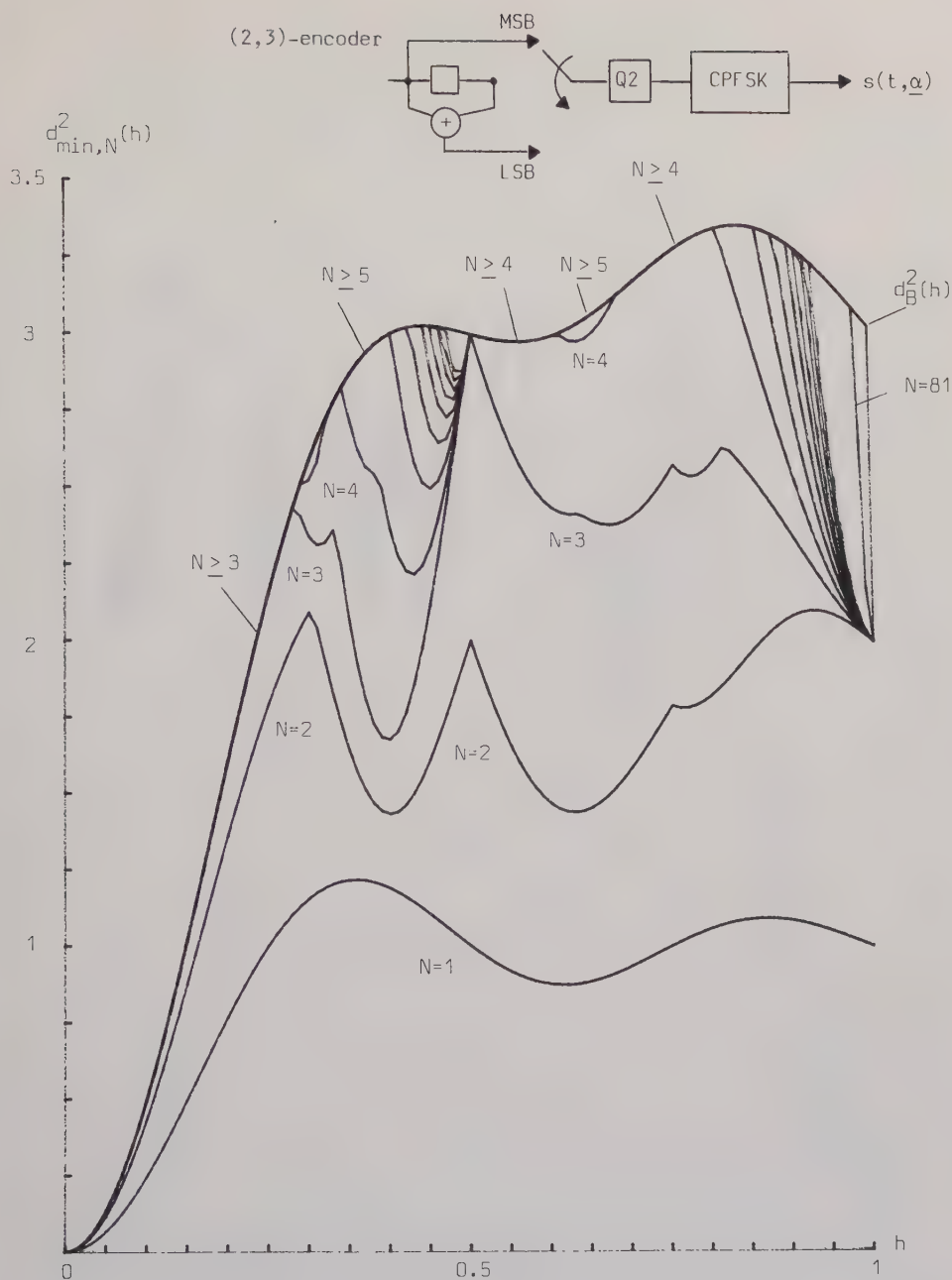


Figure 52. The minimum normalized squared Euclidean distance as a function of h , with N as parameter, for the combination $\{(2,3), Q2\}$ and CPFSK modulation.

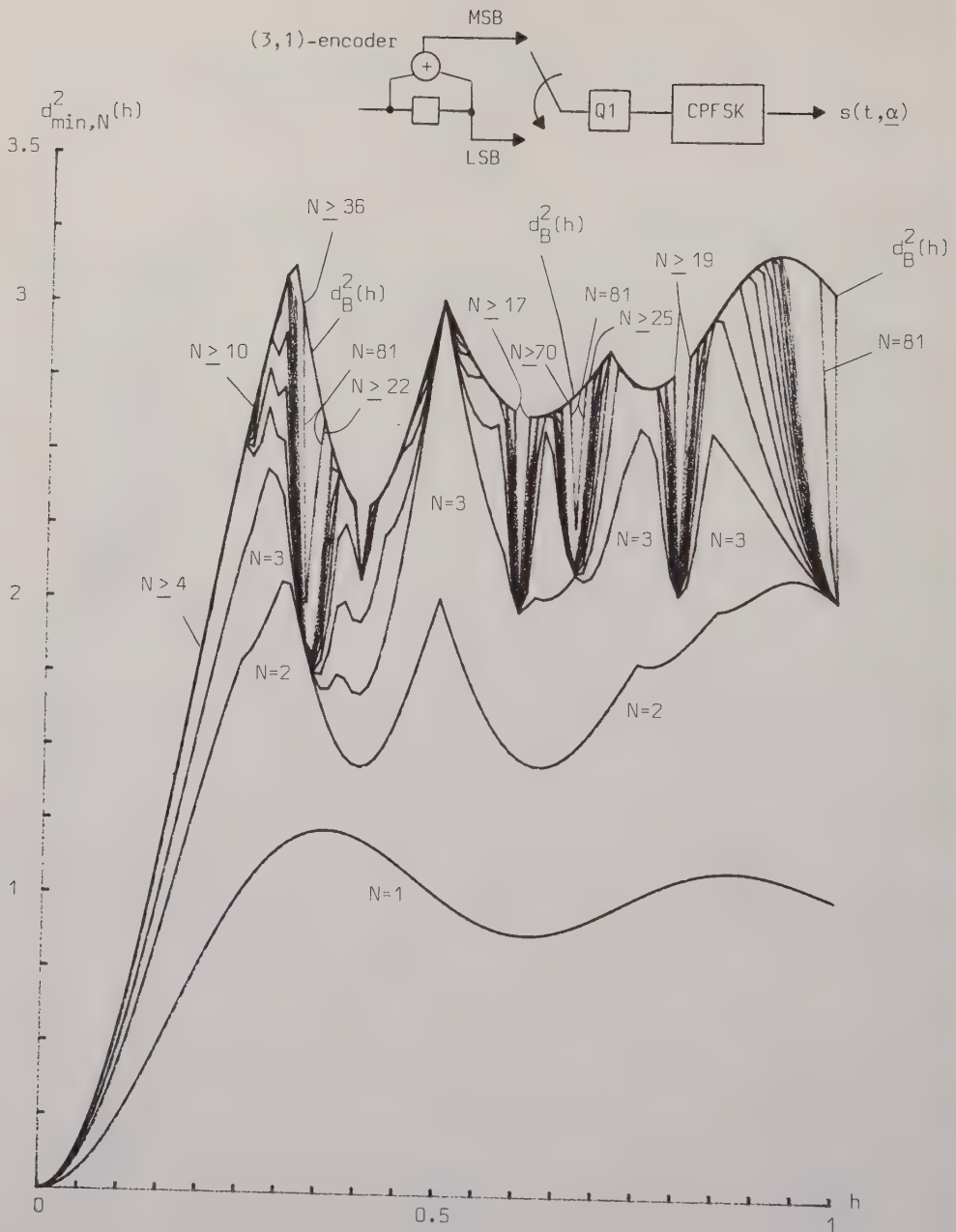


Figure 53. The minimum normalized squared Euclidean distance as a function of h , with N as parameter, for the combination $\{(3,1), Q1\}$ and CPFSK modulation.

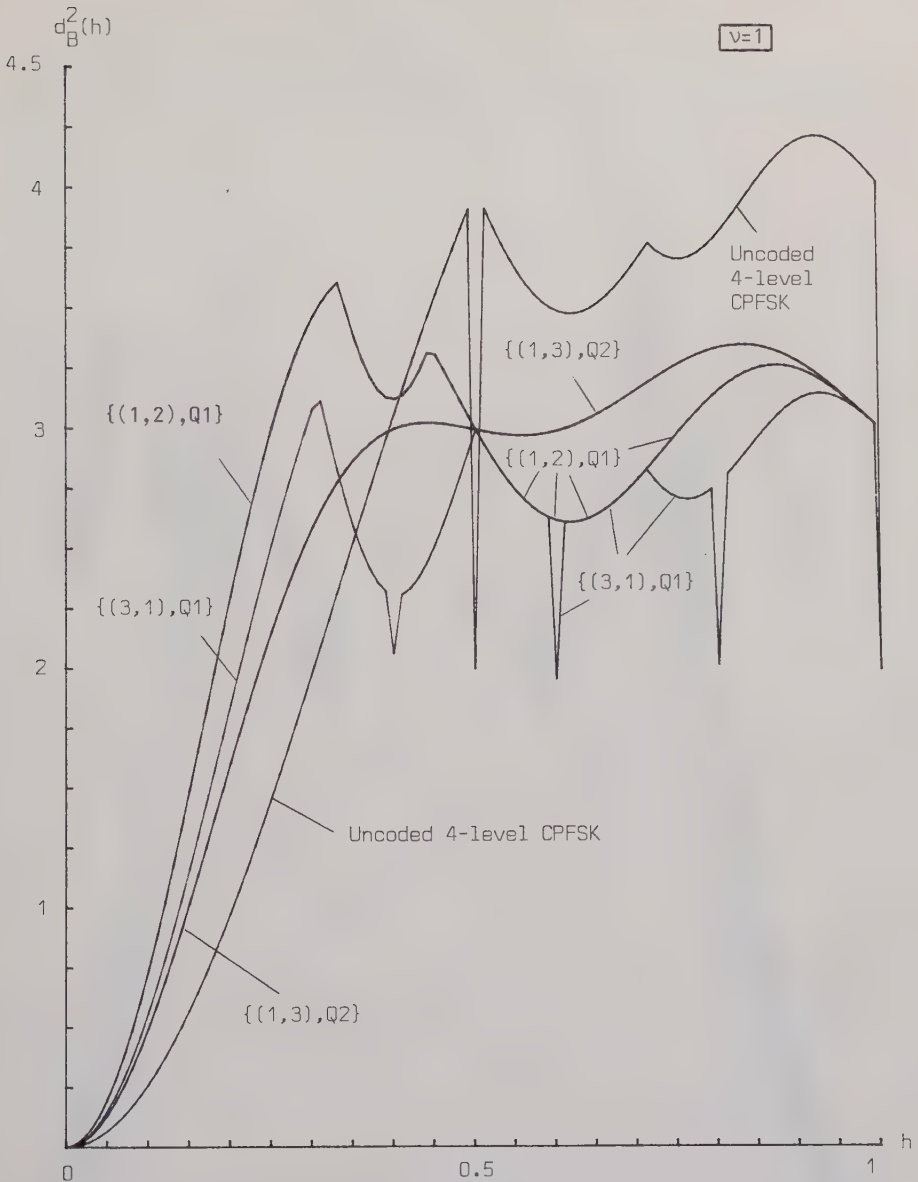
$v=1$ 

Figure 54. The upper bounds (envelopes) in figures 48-53. These bounds are tight for all modulation indices of practical interest.

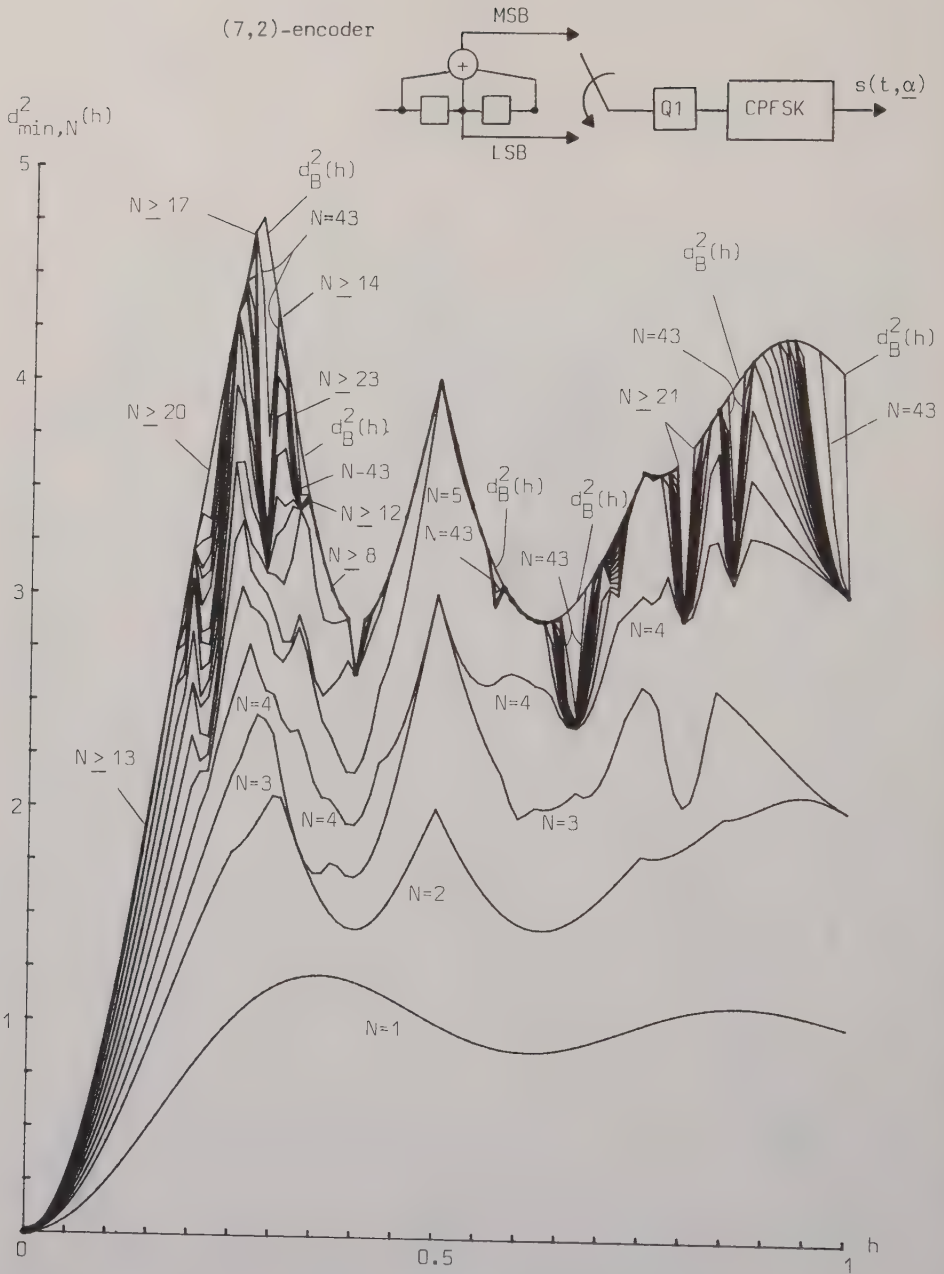


Figure 55. The minimum normalized squared Euclidean distance as a function of h , with N as parameter, for the combination $\{(7,2), Q1\}$ and CPFSK modulation.

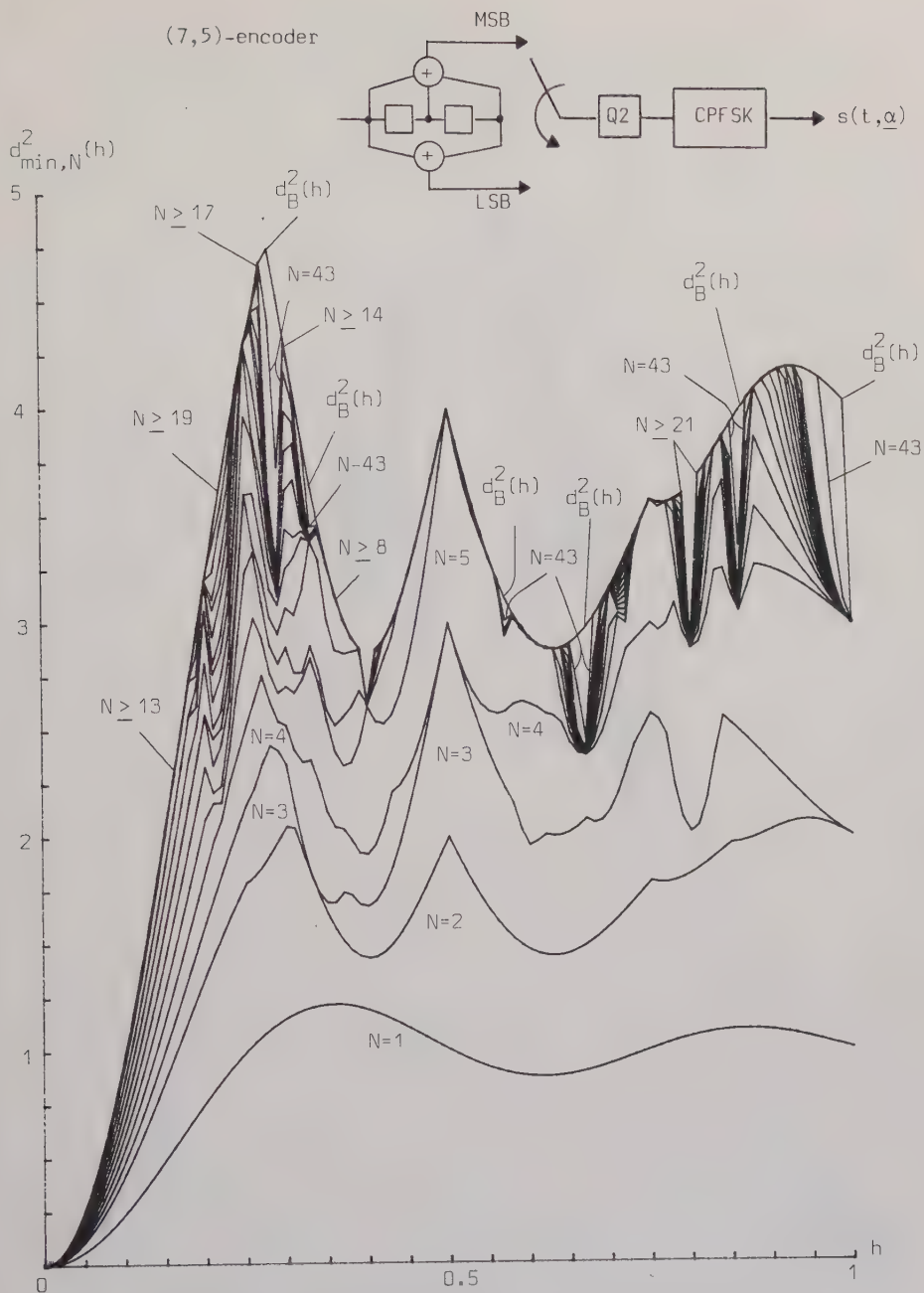


Figure 56. The minimum normalized squared Euclidean distance as a function of h , with N as parameter, for the combination $\{(7,5), Q2\}$ and CPFSK modulation.

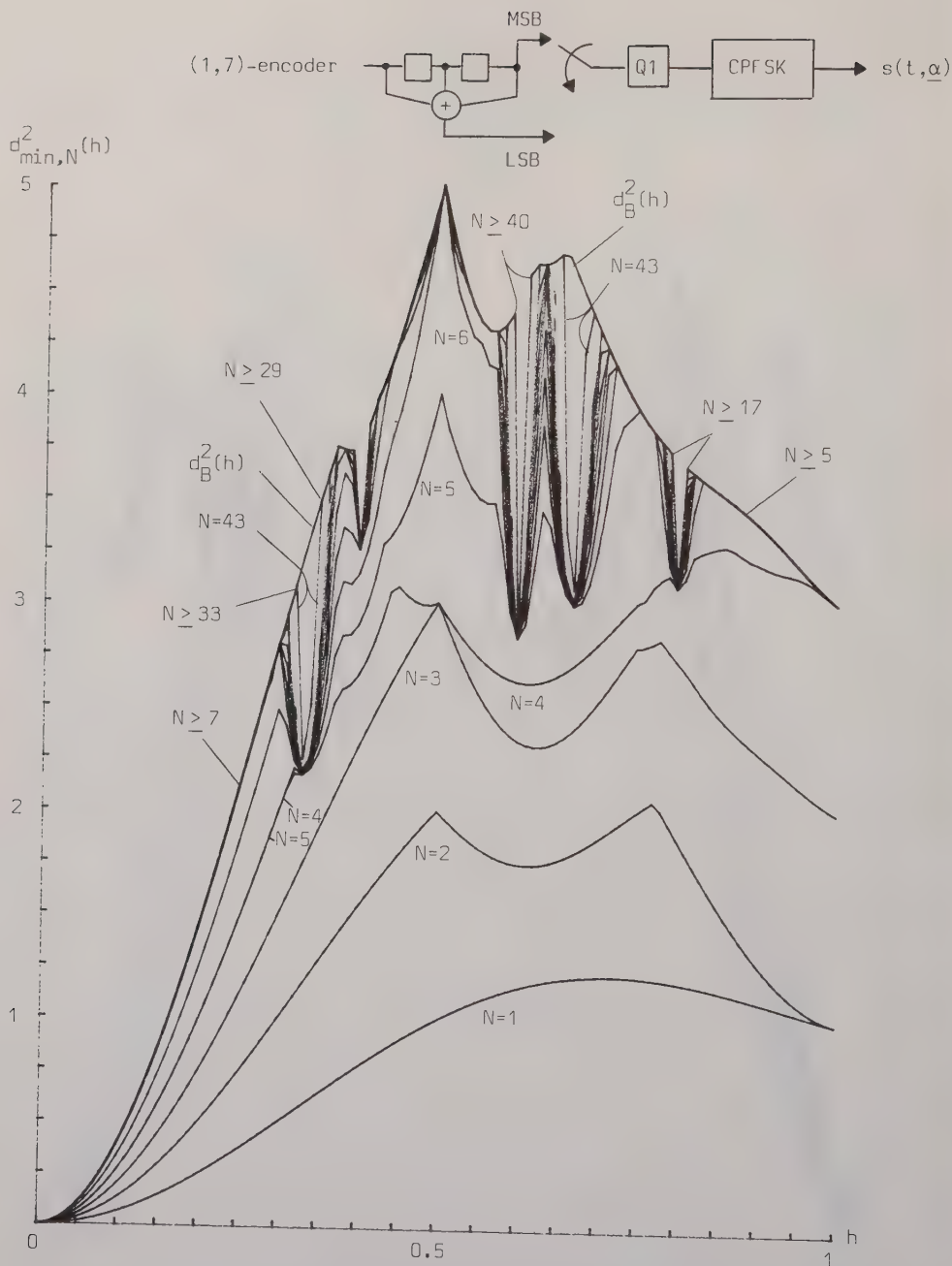


Figure 57. The minimum normalized squared Euclidean distance as a function of h , with N as parameter, for the combination $\{(1,7), Q1\}$ and CPFSK modulation.

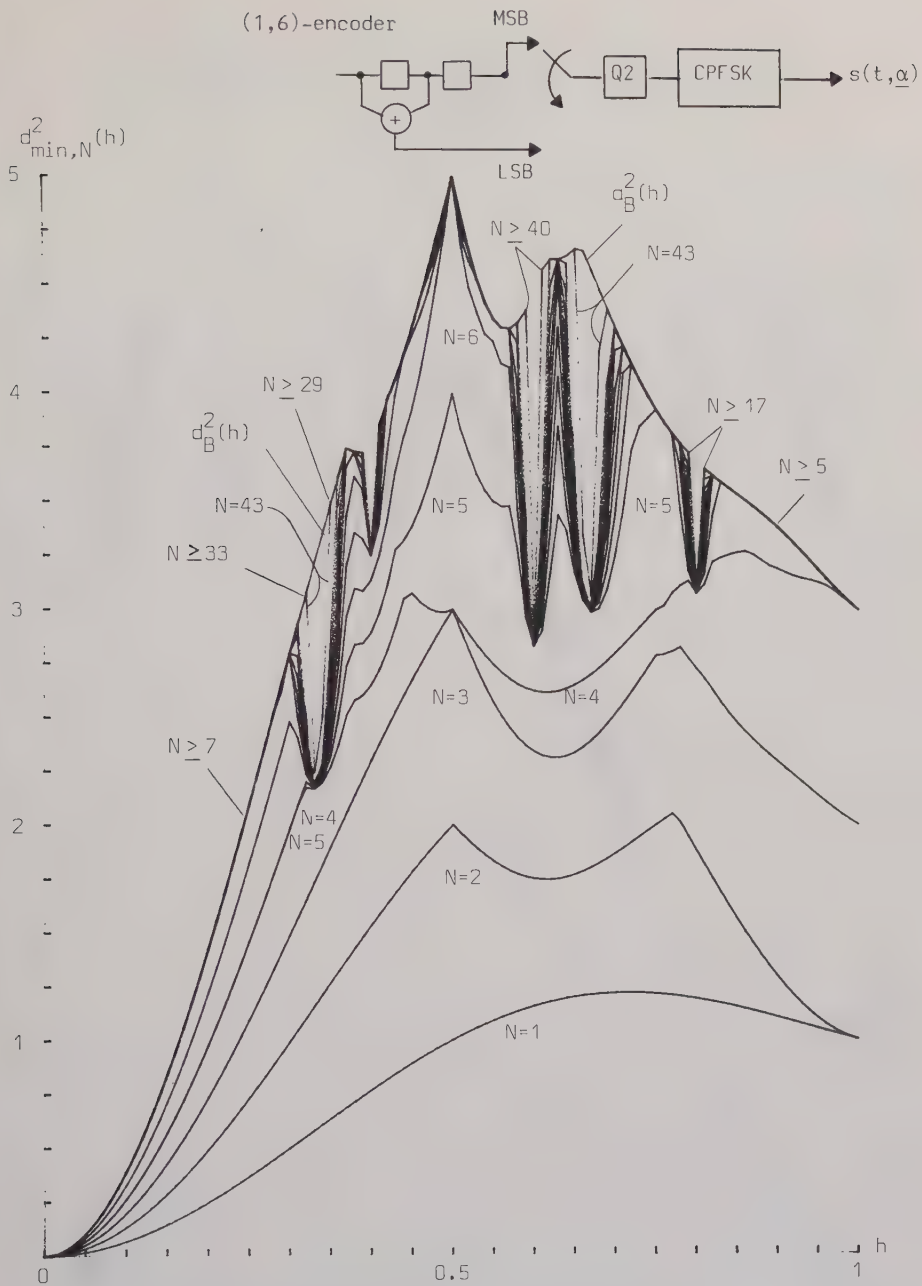


Figure 58. The minimum normalized squared Euclidean distance as a function of h , with N as parameter, for the combination $\{(1,6), Q2\}$ and CPFSK modulation.

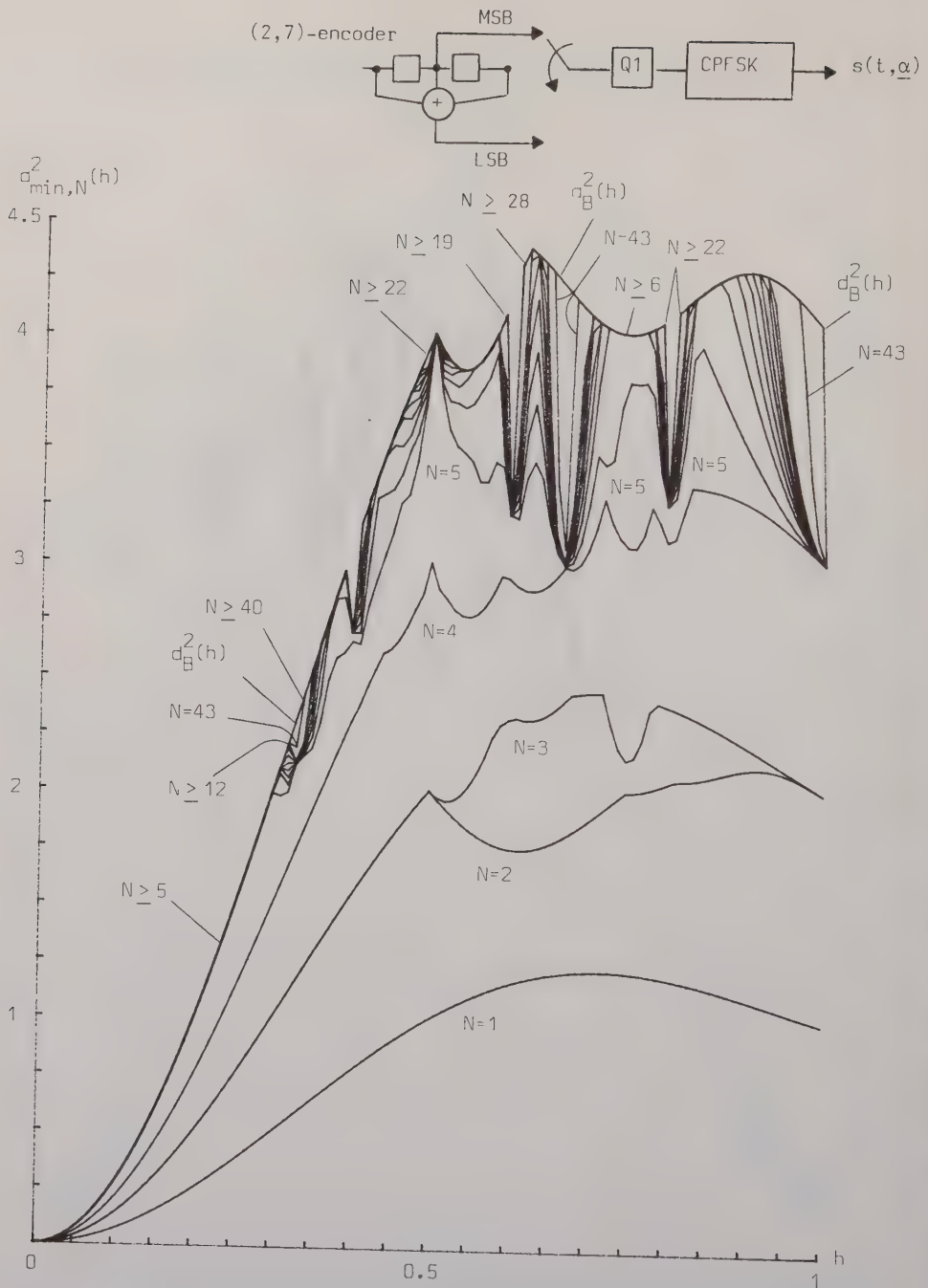


Figure 59. The minimum normalized squared Euclidean distance as a function of h , with N as parameter, for the combination $\{(2,7), Q1\}$ and CPFSK modulation.

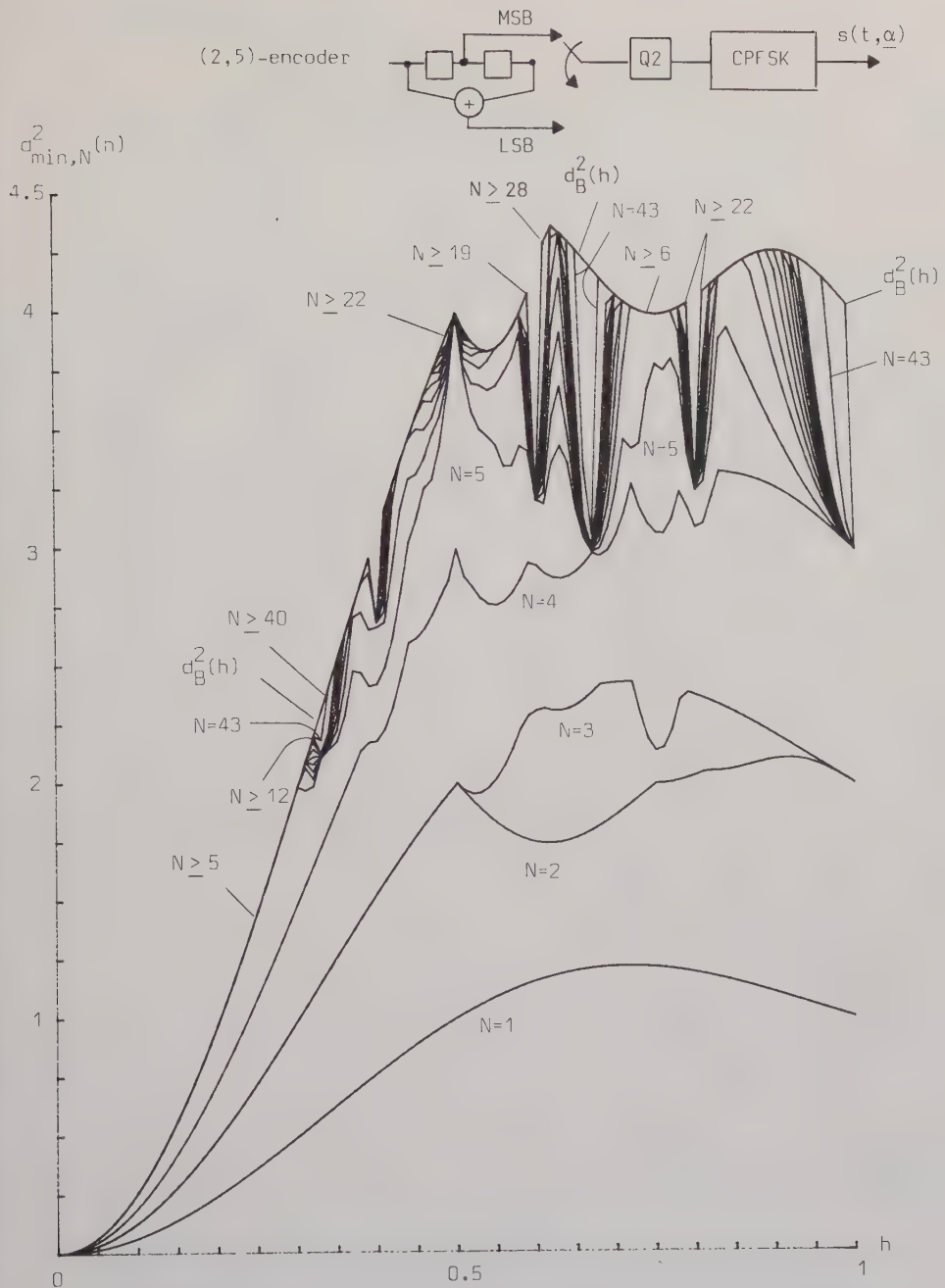


Figure 60. The minimum normalized squared Euclidean distance as a function of h , with N as parameter, for the combination $\{(2,5), Q2\}$ and CPFSK modulation.

Rate 1/2 encoder with 4-level CPFSK

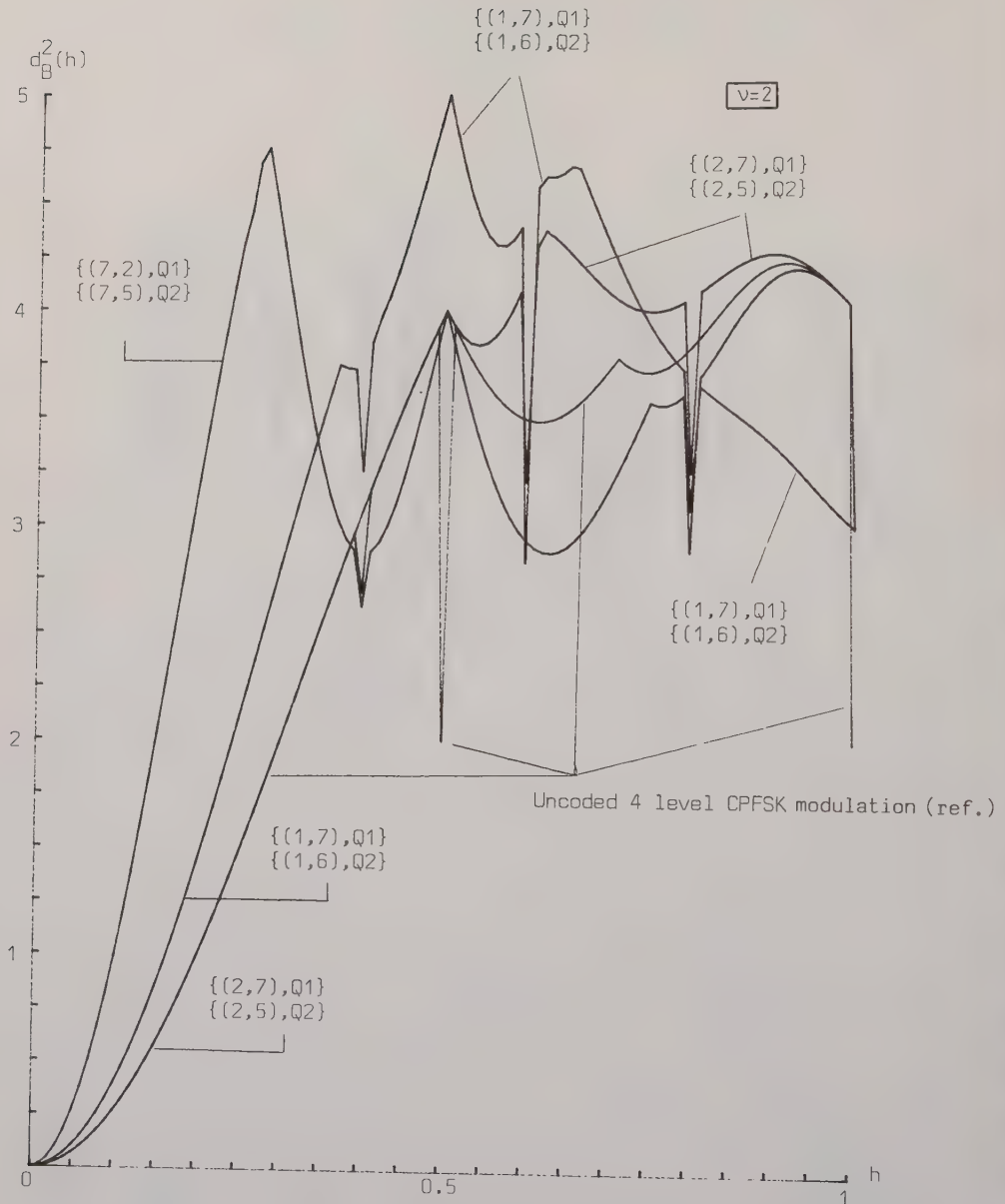
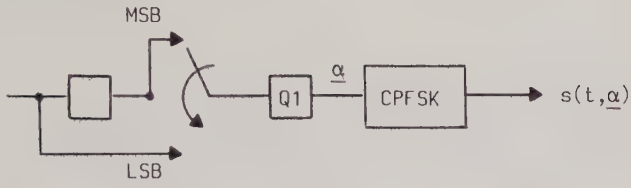


Figure 61. The upper bounds (envelopes) in figures 55-60. These bounds are tight for all modulation indices of practical interest.

(1,2)-encoder



$$\underline{x} = (0)$$

$$\underline{u} = (0, 1, 0) \Rightarrow \underline{\alpha} = (-3, -1, 1)$$

$$\underline{w} = (1, 0, 0) \Rightarrow \underline{\beta} = (-1, 1, -3) \Rightarrow \underline{\gamma} = (-2, -2, 4)$$

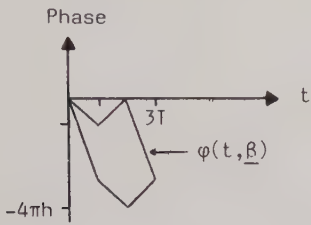
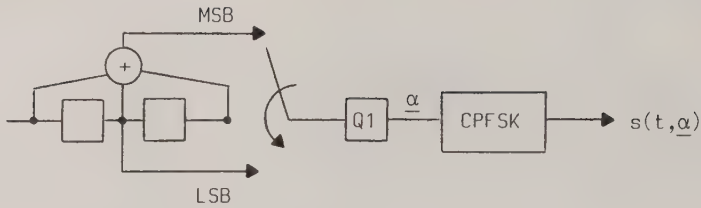


Figure 62. A minimum distance pair when the modulation index $h=1/4$.

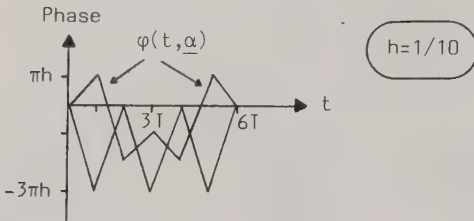
(7,2)-encoder



$$\underline{x} = (0, 1)$$

$$\underline{u} = (0, 0, 1, 1, 1, 0) \Rightarrow \underline{\alpha} = (1, -3, 1, -1, 3, -1)$$

$$\underline{w} = (1, 0, 1, 0, 1, 0) \Rightarrow \underline{\beta} = (-3, 3, -3, 3, -3, 3) \Rightarrow \underline{\gamma} = (4, -6, 4, -4, 6, -4)$$



$$\underline{x} = (0, 0)$$

$$\underline{u} = (0, 0, 1, 1, 1, 1) \Rightarrow \underline{\alpha} = (-3, -3, 1, -1, 3, 3)$$

$$\underline{w} = (1, 0, 1, 0, 1, 1) \Rightarrow \underline{\beta} = (1, 3, -3, 3, -3, -1) \Rightarrow \underline{\gamma} = (-4, -6, 4, -4, 6, 4)$$

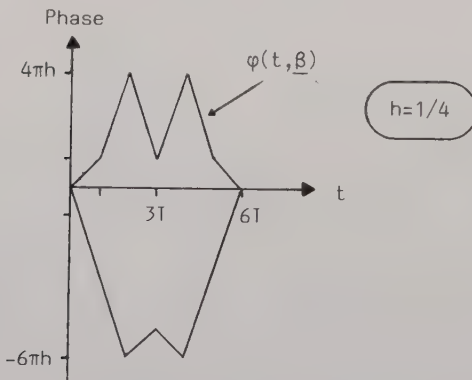
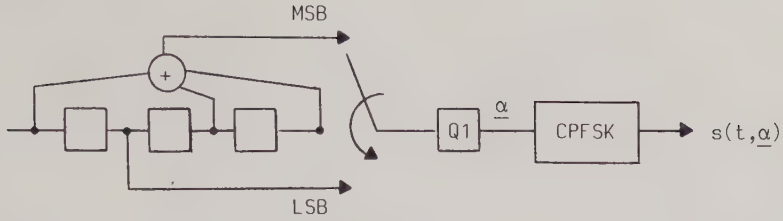


Figure 63. Examples on minimum distance pairs for two different modulation indices.

(13,4)-encoder



$$\underline{x} = (0, 1, 0)$$

$$\underline{u} = (0, 1, 1, 0, 0, 0, 0, 1) \Rightarrow \underline{\alpha} = (1, -3, 3, 3, -3, 1, -3, 1)$$

$$\underline{w} = (1, 0, 0, 0, 1, 0, 0, 1) \Rightarrow \underline{\beta} = (-3, 3, 1, 1, 1, -1, 1, -3) \Rightarrow \underline{\gamma} = (4, -6, 2, 2, -4, 2, -4, 4)$$

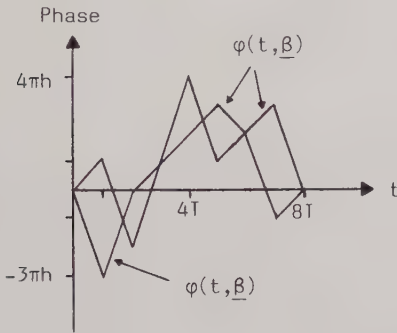
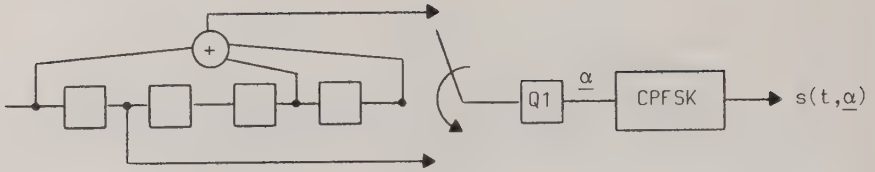


Figure 64. A minimum distance pair when the modulation index $h=1/4$.

(23,10)-encoder



$$\underline{x} = (0, 0, 1, 0)$$

$$\underline{u} = (0, 1, 0, 0, 0, 0) \Rightarrow \underline{\alpha} = (1, -3, -1, -3, 1, 1)$$

$$\underline{w} = (1, 0, 0, 0, 0, 0) \Rightarrow \underline{\beta} = (-3, 3, -3, 1, 1, -3)$$

$$\Rightarrow \underline{\gamma} = (4, -6, 2, -4, 0, 4)$$

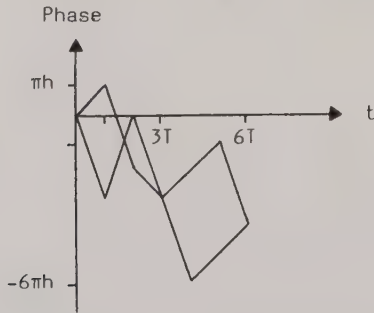


Figure 65. A minimum distance pair when the modulation index $h=1/4$.

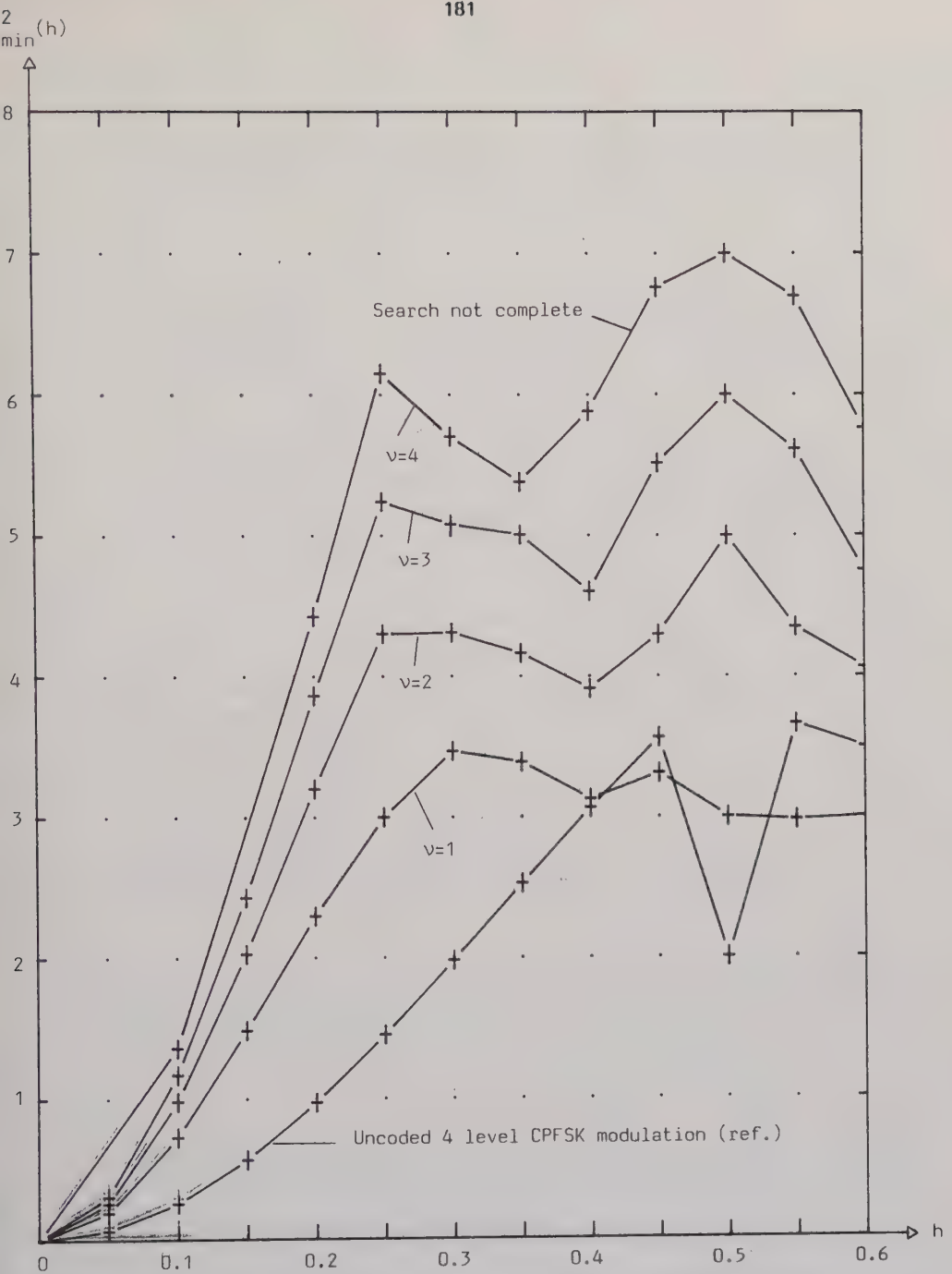


Figure 66. Rate 1/2 convolutional encoder and 4 level CPFSK modulation. The optimum $d_{\min}^2(h)$ for different v and h . See also tables 7-9 and table 2. Note that the connection lines do not contain any numerical results.

$$10 \log_{10}(d_{\min}^2(h)/2)$$

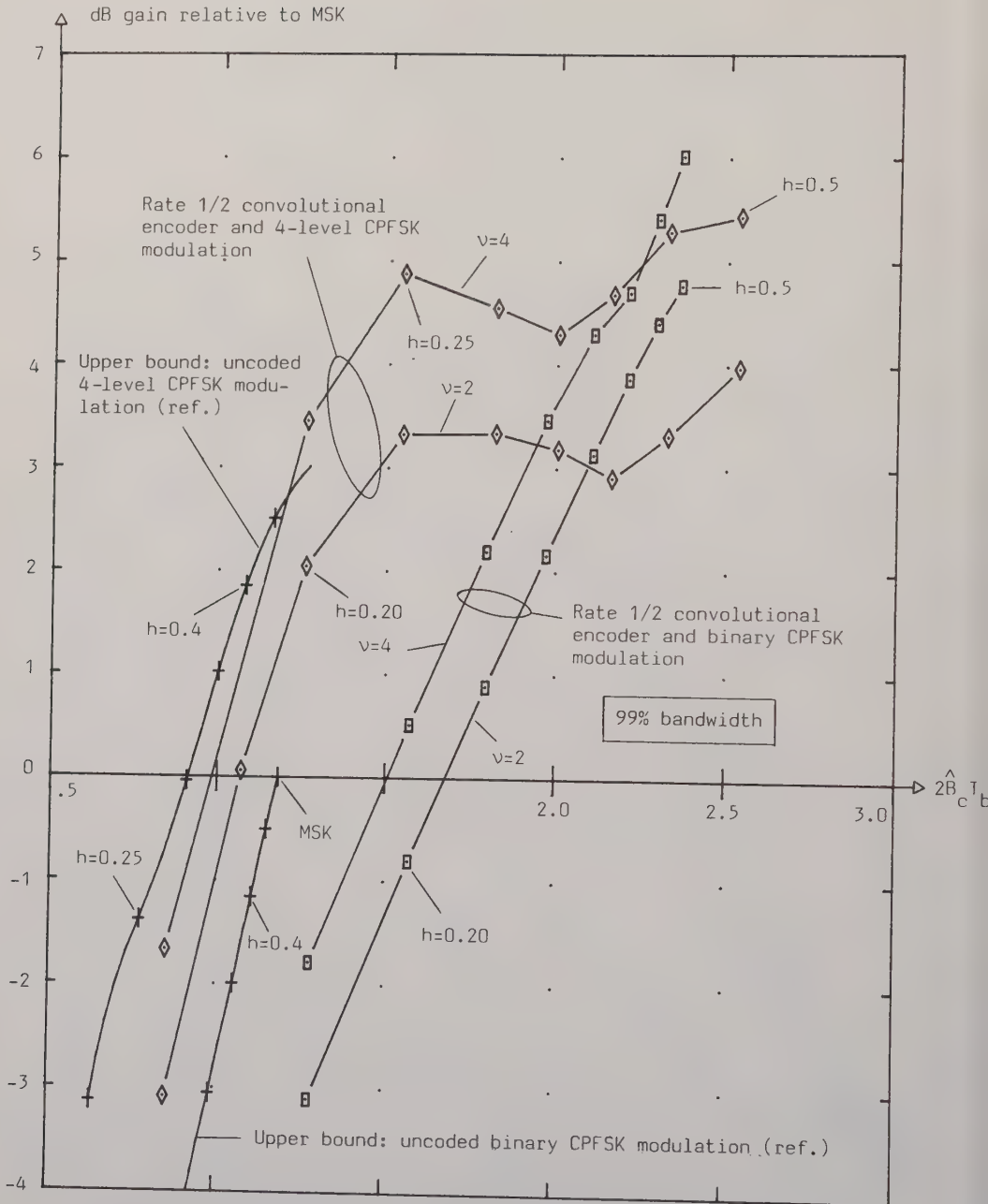


Figure 67. Power-bandwidth tradeoff for some of the coded CPFSK schemes considered in this part. Note that the connection lines do not contain any numerical results.

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ON CODED
CONTINUOUS PHASE MODULATION

PART III

POWER AND BANDWIDTH EFFICIENT CODED
MODULATION SCHEMES

ABSTRACT

Continuous Phase Modulation (CPM) is a class of constant amplitude modulation schemes among which there are modulations with excellent power spectral characteristics. These can be combined with good error probability properties if the receiver performs coherent maximum likelihood sequence detection.

In this part we study the asymptotic error probability (by means of the free Euclidean distance) of signals formed by combining high-rate convolutional codes with multi-level CPM.

In particular we report good short codes for the subclass of schemes consisting of a natural binary mapper for 8 and 16 level continuous phase frequency shift keying (CPFSK) with rate $2/3$ and rate $3/4$ codes.

Code-independent and code-specific upper bounds on the free Euclidean distance have been identified for an interesting subclass of coded multi-level CPM schemes with a natural binary mapper.

Among the results we give schemes with the same asymptotic error probability as minimum shift keying (MSK) but only approximately half the bandwidth (99% in band power). Schemes with roughly the same bandwidth as MSK with a power gain in E_b/N_0 of about 4-6dB are also presented.

A method to reduce the spectral sidelobes while preserving the constant amplitude, the spectral mainlobe and approximately the asymptotic error probability is also introduced.

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I. INTRODUCTION

During the last few years a large number of constant amplitude digital modulation schemes have been proposed and analysed in the literature. Many of these schemes fit in as special cases in the continuous phase modulation (CPM) class [1]-[2]. Memory is introduced into the signal by means of continuous phase and also by means of partial response smoothing (correlative encoding). Uncoded partial response CPM yield good combined power and bandwidth efficiency [1]-[2].

From the analysis of uncoded multilevel continuous phase frequency shift keying (CPFSK) in [1] (and in principle the same holds for uncoded multilevel partial response CPM [2] with reasonable smoothing) it is evident that if certain information carrying continuous phase functions (corresponding to certain sequences of input symbols to the modulator) are "avoided", the distance properties of the resulting signals can be significantly increased. Thus a coding rule is introduced for avoiding certain sequences in the uncoded CPFSK scheme. This code, as it turns out, can sometimes have quite a high rate. Thus the penalty in energy per information bit and bandwidth expansion is rather limited compared to the large increase in minimum distance (yielding a corresponding improvement in asymptotic error probability).

In this part, combined convolutional coding and multilevel CPM is considered. Low modulation indices are particularly interesting since this yields narrowband systems. We will typically deal with 8-level CPM (and up). The over all code rate with 8-level schemes is $2/3$, i.e. 2 information bits per 8-level channel symbol. 16-level CPM with rate $3/4$ codes and 32-level CPM with rate $4/5$ codes are also studied.

Because of the gigantic number of combined CPM schemes and codes, we will search for good (in some cases locally optimal) schemes in a carefully designed subclass. From [1]-[2], one realizes that, to begin with, only the least significant bits of the natural binary mapper input word needs coding. This significantly simplifies the code selection.

In this part results on coded multilevel CPFSK schemes are presented, and tables of good schemes are given. These schemes combine power and bandwidth efficiency with constant amplitude. Significant improvements compared to uncoded CPM and multi-h schemes [7]-[8], are obtained, both in terms of bandwidth conservation and power gains.

We will only consider ideally coherent transmission over an additive white Gaussian noise (AWGN) channel.

The receiver is assumed to be an ideal maximum likelihood receiver. The price for good performance is receiver complexity. The most important parameters which contribute to this complexity will be identified and discussed below.

Independent work on rate $2/3$ convolutional encoders combined with 8-level CPFSK can be found in [4]. In [5] rate $1/2$ encoders combined with 4-level CPFSK, 2RC and 3RC respectively, are studied. See also [6].

The underlying idea of combining a convolutional code and a multilevel modem and search for schemes with good Euclidean distance properties have previously been considered by Ungerboeck [3]. See also [12]–[13]. A special kind of "coded" CPFSK and CPM schemes are the so called multi-h schemes, [7]–[8]. These are CPM schemes where the modulation index varies periodically with time.

In this part we study a restricted class of high rate convolutional encoders. The studied encoders have $(n-1)$ inputs ($k=n-1$) and n outputs, $n = 3, 4, 5, 6$. Furthermore, the first $(k-1)$ bits in each block (of length k bits) are not coded at all

$$v_{j,m} = u_{j,m}, \quad m = 1, 2, \dots, k-1; \quad \text{all } j \quad (1)$$

The only coding used is a conventional (in general nonsystematic) rate $1/2$ convolutional encoder which encodes the data symbol sequence

$\dots, u_{-1,k}, u_{0,k}, u_{1,k}, \dots$ to produce the two coded symbol sequences $v_{-1,n-1}, v_{0,n-1}, v_{1,n-1}, \dots$ and $\dots, v_{-1,n}, v_{0,n}, v_{1,n}, \dots$ respectively. Formally this can be written as

$$v_{j,m} = \sum_{i=0}^v g_{m,k}(i) u_{j-i,k}; \quad m = n-1, n, \quad \text{all } j \quad (2)$$

When a specific rate $1/2$ convolutional encoder is referred to, an **octal** representation of the two polynomials $g_{n-1,k}(i)$ and $g_{n,k}(i)$ is used. The encoder described above has $(n-1)$ inputs and n outputs, therefore the rate of the encoder is $R_c = (n-1)/n$ bits/coded symbol. Furthermore, this encoder has $v_1 = v_2 = \dots = v_{k-1} = 0$ and $v_k = v$.

The rate of the mapper is in this part matched to the number of outputs of the encoder in such a way that $R_m = n$. It is also assumed that the 2^n -level natural binary mapping rule is used. Thus the channel encoding is a conventional rate $1/2$ convolutional encoder on the least significant sym-

bols in the mapping rule. The overall rate is $R = R_c R_m = (n-1)$ bits/channel symbol. Fig. 1 shows some examples on combinations of encoders and mappers as described above. In fig. 1a $n=3$ and in fig. 1b $n=4$.

In this part it is assumed that the frequency pulse is

$$g(t) = \begin{cases} [1 - \cos(\pi t / \epsilon T)] / 4T, & 0 < t < \epsilon T \\ 1/2T, & \epsilon T \leq t \leq T \\ [1 + \cos(\pi(t-T) / \epsilon T)] / 4T, & T < t < (1+\epsilon)T \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

with $0 \leq \epsilon \leq 1$. When $\epsilon=0$ and when $\epsilon=1$ we have the known CPFSK scheme and 2RC scheme respectively. Thus, eq. (3) defines a class of frequency pulses in between CPFSK and 2RC, and a certain degree of smoothness and partial response can be controlled by the parameter ϵ , see figs. 2-3. Using CPFSK modulation ($\epsilon=0$) we are searching for the optimal noncatastrophic rate $1/2$ convolutional encoders such that, for given v and h , $d_{\min}^2$ is maximized. Formally this can be written as

$$\max_{\left\{ \begin{array}{c} \text{noncatastrophic} \\ \text{rate } 1/2 \\ \text{encoders} \end{array} \right\}} d_{\min}^2 \quad (4)$$

Some of the found optimal encoders are then used together with a frequency pulse with a small ϵ , $\epsilon > 0$. The idea is to create a modulation scheme which has roughly the same free Euclidean distance as the corresponding coded CPFSK scheme but with considerably smaller sidelobes in the spectrum due to the introduced smoothing and partial response.

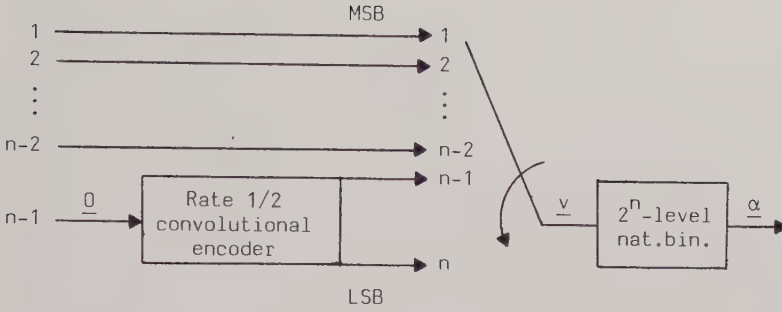
II. ENCODER-INDEPENDENT UPPER BOUNDS ON $d_{\min}^2$

From Part I it is known that the Euclidean distance between any two transmitted signals $s(t, \underline{\alpha})$ and $s(t, \underline{\beta})$ is a function of the difference sequence $\underline{\gamma}$. Therefore, to calculate an upper bound on $d_{\min}^2$ only the difference sequence corresponding to a specific phase and state merger has to be known. By feeding the convolutional encoders studied in this part with zeroes in input number k it is possible to generate phase and state mergers corresponding to the difference sequences $\underline{\gamma} = (\dots, 0, 8\ell, -8\ell, 0, \dots)$, $\ell = 1, 2, \dots, 2^{n-2}-1$. This, and eq. (5), is proved at the end of this section. It should be noted that these phase and state mergers are independent of the actual rate $1/2$ convolutional encoder used. Thus, encoder-independent upper bounds on $d_{\min}^2$ can be calculated based on these phase and state mergers. Taking the minimum of these $(2^{n-2}-1)$ upper bounds, and assuming CPFSK modulation, yields the encoder-independent upper bound

$$d_{\min}^2(h) \leq 2(n-1) \cdot \min_{1 \leq \ell \leq A} \left[1 - \frac{\sin(\ell 8\pi h)}{\ell 8\pi h} \right] \quad (5)$$

where $A = 2^{n-2}-1$. Fig. 4 shows this upper bound with n as a parameter. Two questions arise naturally. Are the upper bounds tight? If they are tight, which rate $(n-1)/n$ convolutional encoders combined with 2^n -level CPFSK modulation reaches this bound? In particular we are interested in small modulation indices ($h \lesssim 1/4$). It should be noted that as we increase n , the number of encoder-independent weak modulation indices are also increased and this is due to the distance properties of the uncoded 2^{n-2} level CPFSK scheme, see proof below.

Proof of equation (5):



If the convolutional encoder given above, $n \geq 3$, is fed with the all zero sequence (0) in its $(n-1)$:th input, the channel symbol sequence α_j becomes (see eq. (2) in Part I)

$$\alpha_j = \sum_{i=0}^{n-3} v_{jn+i} 2^{(n-i)} - M+1 = 4 \left(\sum_{i=0}^{n-3} v_{jn+i} 2^{(n-2+i)} - 2^{n-2+1} \right) - 3 \quad (6)$$

Now use the notation $\tilde{s} = \tilde{R}_m = n-2$ and $\tilde{M} = 2^{n-2}$. Then eq. (6) can be written as

$$\alpha_j = 4 \left(\sum_{i=0}^{\tilde{s}-1} v_{jn+i} 2^{(\tilde{s}-i)} - \tilde{M} + 1 \right) - 3 = 4\tilde{\alpha}_j - 3 \quad (7)$$

where $\tilde{\alpha}_j$ is recognized as a channel symbol generated from an uncoded $\tilde{M}$ -level scheme.

Thus, any difference-symbol $\gamma_j = \alpha_j - \beta_j$ can be written as

$$\gamma_j = 4\tilde{\gamma}_j \quad (8)$$

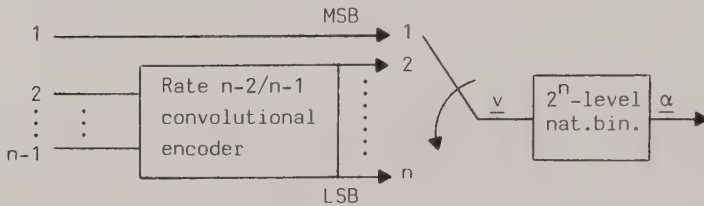
where $\tilde{\gamma}_j$ is a difference-symbol generated from an uncoded $\tilde{M}$ -level scheme. Since $\tilde{\gamma}_j \in \{0, \pm 2, \pm 4, \pm 6, \dots, \pm 2(\tilde{M}-1)\}$ all j , it is clear that the difference-sequences $\tilde{\gamma} = (\dots, 0, 0, 0, \tilde{\gamma}_0, -\tilde{\gamma}_0, 0, 0, 0, \dots)$ exists, where $\tilde{\gamma}_0 \in \{2, 4, 6, \dots, 2(\tilde{M}-1)\}$. Therefore, the difference-sequences $\underline{\gamma} = (\dots, 0, 0, 0, 8\ell, -8\ell, 0, 0, 0, \dots)$, $\ell = 1, 2, \dots, 2^{n-2}-1$, must also exist. Each such $\underline{\gamma}$ -sequence produce a phase and state merger, and can therefore be used to calculate an upper bound on $d_{\min}^2(h)$. Taking the minimum of these $(2^{n-2}-1)$ upper bounds and assuming CPFSK modulation (use eq. (26) in Part I) yield eq. (5).

It should be noted that eq. (8) also implies that

$$d_{\min}^2(h) \leq \frac{n-1}{n-2} \tilde{d}_{\min}^2(4h) \quad (9)$$

where $\tilde{d}_{\min}^2(\tilde{h})$ equals the normalized squared free Euclidean distance for the uncoded 2^{n-2} -level scheme with modulation index $\tilde{h}$. The factor $(n-1)/(n-2)$ is due to the differences in the over-all rate.

The technique used in the proof above to obtain encoder-independent upper bounds, can easily be applied to other classes of convolutional encoders. An example of such a class is given below ($n \geq 3$).



III. NUMERICAL RESULTS

For the case of rate 2/3 convolutional encoders with 8-level CPFSK modulation, the encoder consists of an uncoded MSB and a rate 1/2 code on the remaining bits, see example in fig. 1a. The optimum (largest $d_{\min}^2(h)$) rate 1/2 convolutional encoders have been found when $1 \leq v \leq 4$. Some of these rate 1/2 encoders are then used in rate 3/4 and rate 4/5 convolutional encoders, and $d_{\min}^2(h)$ is calculated for these encoders when combined with 16-level and 32-level CPFSK modulation, respectively. Low modulation indices are particularly interesting. The specific modulation indices studied are $h = 2/22, 2/21, \dots, 2/9, 2/8$. These choices of h -values are dependent on the algorithm used when calculating $d_{\min}^2(h)$, see Appendix A. The numerical results are presented in tables 1-8. These tables give the information {rate 1/2 encoder, N, N_m } where N is the smallest value of N for which $d_{\min, N}^2 = d_{\min}^2$ and N_m is the length of the shortest minimum distance pair (phase and state merger with minimum Euclidean distance). Furthermore the $d_{\min}^2(h)$ -values given in these tables are underlined if the encoder-independent upper bound on $d_{\min}^2(h)$ is reached. Note that if the upper bound is reached it is not possible to increase $d_{\min}^2(h)$ regardless of how many delay elements we use in the rate 1/2 encoder.

III.1 Rate 2/3 Convolutional Encoders and 8-level CPFSK Modulation

When $v=1$ the optimal rate 1/2 convolutional encoders are given in table 1 together with the corresponding optimum $d_{\min}^2(h)$. As an example, when $h=1/10$ it is seen that the optimum $d_{\min}^2(1/10) = 0.9826$, and there are four optimal encoders for this h -value. These four encoders are shown in fig. 5. It is also seen that for $h=1/4$ the upper bound is reached. But since $h=1/4$ is an encoder-independent weak modulation index this is not so exciting.

When $v=2$ the optimal rate 1/2 convolutional encoders are given in table 2 together with the corresponding optimum $d_{\min}^2(h)$. For small modulation indices, the (5,2)-encoder and the (7,2)-encoder are optimal, see fig. 6. The $d_{\min}^2(h)$ -values achieved are significantly larger than the corresponding values for the $v=1$ case and the upper bound is reached for $h=2/9$, $d_{\min}^2(2/9) = 4.4604$. For this h -value there are eight optimal encoders.

When $v=3$ the optimal rate 1/2 convolutional encoders are given in table 3 together with the corresponding optimum $d_{\min}^2(h)$. For small modulation indices there are four optimal encoders, see fig. 7. The $d_{\min}^2(h)$ -values achieved are significantly larger than the corresponding values for the $v=2$

case and the upper bound is reached for $h=1/5$, $d_{\min}^2(1/5) = 4.7568$. For this h -value there are eight optimal encoders.

When $v=4$ the calculations are getting heavy. As a consequence, a complete search has only been done for the modulation indices $h = 1/11$, $1/6$ and $2/11$. For all the other modulation indices the (33,4)-encoder has been used and the corresponding $d_{\min}^2(h)$ -values have been calculated, see table 4 and fig. 8. Compared to table 3 ($v=3$) it is seen that the achieved $d_{\min}^2(h)$ -values are significantly increased. The upper bound is now reached for the modulation indices $h = 2/15$, $1/7$, $2/13$, $1/6$ and $h=2/11$.

III.2 Rate 3/4 and Rate 4/5 Convolutional Encoders

The modulation schemes considered in this section are rate 3/4 and rate 4/5 convolutional encoders combined with 16 and 32-level CPFSK modulation respectively. No search for the optimal rate 1/2 encoders has been performed. Instead, some of the optimal encoders found in section III.1 are used. Specifically, the (2,1), (5,2), (15,2) and (33,4)-encoders are used. The calculated $d_{\min}^2(h)$ -values are given in tables 5 to 8. It is seen that there is quite a significant increase in the $d_{\min}^2(h)$ -values when v is increased. Furthermore, for small modulation indices ($h \leq 2/17$) the upper bound has not been reached when four or fewer delay elements are used in the encoder.

III.3 Summary of Numerical Results

Very good schemes of combined rate $(n-1)/n$ convolutional encoders and 2^n -level CPFSK modulation, $n = 3, 4, 5$, have been found when the modulation index is small ($h \leq 1/4$). Examples of good rate 1/2 encoders are the (2,1)-encoder, the (5,2)-encoder, the (15,2)-encoder and the (33,4)-encoder. These encoders are shown in fig. 9 assuming $n=4$.

When using the four rate 1/2 encoders given above, with $n=3$, the shortest minimum distance pairs have been found (small h), see figs. 10-13. The corresponding $\underline{\gamma}$ -sequences are $\underline{\gamma} = (-4, 6, -2)$, $\underline{\gamma} = (-4, 6, -6, 4)$, $\underline{\gamma} = (-4, 8, -6, 6, -4)$ and $\underline{\gamma} = (-4, 4, 2, 0, -2, -4, 4)$, respectively. If we use these $\underline{\gamma}$ -sequences to construct upper bounds on $d_{\min}^2(h)$ we obtain (CPFSK modulation)

$$d_{\min}^2(h) \leq (n-1) \left(3 - \frac{4}{3} \frac{\sin(2\pi h)}{2\pi h} - \frac{5}{3} \frac{\sin(4\pi h)}{4\pi h} \right) ; \quad (2,1)\text{-encoder} \quad (10)$$

$$d_{\min}^2(h) \leq (n-1) \left(4 - \frac{2}{3} \frac{\sin(2\pi h)}{2\pi h} - \frac{10}{3} \frac{\sin(4\pi h)}{4\pi h} \right) ; \quad (5,2)\text{-encoder} \quad (11)$$

$$d_{\min}^2(h) \leq (n-1) \left(5 - \frac{2}{3} \frac{\sin(2\pi h)}{2\pi h} - \frac{13}{3} \frac{\sin(4\pi h)}{4\pi h} \right) ; \quad (15,2)\text{-encoder} \quad (12)$$

$$d_{\min}^2(h) \leq (n-1) \left(7 - 2 \frac{\sin(2\pi h)}{2\pi h} - 4 \frac{\sin(4\pi h)}{4\pi h} - \cos(2\pi h) \right) ; \quad (33,4)\text{-encoder} \quad (13)$$

Note that a specific γ -sequence found when $n=3$ does also exist when $n > 3$ if the same rate $1/2$ encoder is used. This is most easily seen by feeding the first $(n-3)$ inputs with zeroes. The channel symbol α_j then becomes (see eq. (2) in Part I)

$$\alpha_j = \sum_{i=n-3}^{n-1} v_{jn+i} \cdot 2^{(n-i)-2^{n+1}} = \sum_{i=0}^{3-1} v_{(j+1)n-3+i} \cdot 2^{(3-i)-2^3+1+2^3-2^n} = \tilde{\alpha}_j + 8 \cdot 2^n \quad (14)$$

where $\tilde{\alpha}_j$ is the channel symbol associated with the corresponding rate $2/3$ encoder and 8-level scheme. Thus, eqs. (10)-(13) are upper bounds on $d_{\min}^2(h)$ for all $n \geq 3$.

Some of the encoder-dependent upper bounds on $d_{\min}^2(h)$, given in eqs. (10)-(13), are shown in figs. 14-15 ($\epsilon=0$) when $n=3$ and $n=4$, respectively. These figures do also show the corresponding encoder-independent upper bound (eq. (5)) and uncoded upper bound (ref.). Since many of the upper bounds in figs. 14-15 are tight (CPFSK case), these figures actually show $d_{\min}^2(h)$ for some specific coded CPFSK schemes.

Let us still use the four rate $1/2$ encoders given above. We are now interested in how $d_{\min}^2(h)$ is affected by ϵ . That is, we replace the CPFSK frequency pulse with another frequency pulse, according to eq. (3), with ϵ -values in the interval $0 < \epsilon \leq 1$. As is seen in figs. 14-15 the upper bounds on $d_{\min}^2(h)$ do not change very much as long as $0 \leq \epsilon \leq 1/2$. For ϵ -

values in the interval $1/2 \leq \epsilon \leq 1$ the difference is larger. Thus we can expect that for small ϵ -values $d_{\min}^2(h)$ is roughly the same as for the corresponding coded CPFSK case. For the uncoded case with $\epsilon=1$, the bounds in figs. 14-15 are tight (see 2RC in [2]).

IV. POWER AND BANDWIDTH EFFICIENT CODED CPM SCHEMES

Fig. 16 shows examples of estimated power spectral densities when a rate $2/3$ convolutional encoder and 8-level CPM are used. The modulation index in this figure is $h=0.091$ ($\approx 1/11$), and the reduction of the sidelobes with increasing ϵ is clearly demonstrated. Fig. 17 shows estimated power spectral densities for two interesting coded CPFSK schemes. The power spectra for conventional uncoded MSK is also given in this figure as a reference. It is seen in this figure that the rate $2/3$ (33,4)-encoder combined with 8-level CPFSK and $h=1/11$ yields a spectrum which is much better than the MSK spectrum. It should be observed, that this coded scheme has $d_{\min}^2 = 2.1615$ which is slightly larger than $d_{\min}^2$ for MSK which equals 2. Thus, this coded CPFSK scheme saves bandwidth compared to MSK. It is also seen in fig. 17 that the rate $3/4$ (33,4)-encoder combined with 16-level CPFSK and $h=2/15$ yields a spectrum which also is better than the MSK spectrum. It should be observed that this coded scheme has $d_{\min}^2 = 5.64$. Thus, this coded CPFSK scheme saves both power and bandwidth relative MSK.

Note that still more spectrally efficient schemes easily can be obtained by using a small ϵ , $\epsilon > 0$ instead of using $\epsilon=0$ (CPFSK). If $0 \leq \epsilon \leq 1/2$ it is expected that $d_{\min}^2$ will not change significantly.

Fig. 18 shows the estimated bandwidth as a function of the modulation index for some of the coded CPFSK schemes considered in this part. The bandwidth for the corresponding uncoded CPFSK schemes is also given in this figure as references.

Figs. 19-21 show power-bandwidth tradeoffs for some of the schemes considered in this part. On the vertical axis is the gain in $d_{\min}^2$ compared to MSK and on the horizontal axis is the estimated normalized double-sided bandwidth (99%). Thus the asymptotic error performance compared with MSK versus bandwidth is given in these figures. It should be noted that the connection lines between schemes with different h in these figures do not contain any numerical results. Fig. 19 considers schemes consisting of the rate $2/3$ encoders given in tables 1-4 combined with 8-level CPFSK modulation. Specific schemes are shown as rectangles or diamonds in figs. 19-21. Note that coded schemes with the same h -value have the same estimated bandwidth for all v -values. It is seen in fig. 19 that the coded schemes become more power efficient with increasing v , until the encoder-independent upper bound is reached. For modulation indices $h \leq 1/8$ this upper bound has not been reached but it seems likely that it will be reached if more

delay elements are used in the rate $1/2$ encoder. Fig. 20 considers schemes consisting of the rate $3/4$ encoders given in tables 5-6 combined with 16-level CPFSK modulation. These schemes are also becoming more power efficient with increasing v , until the encoder-independent upper bound is reached.

Fig. 21 considers the best schemes found in this part. It is quite obvious from this figure that schemes have been found which are much better than the MSK scheme both in terms of asymptotic error performance and in terms of bandwidth. As an example of such a scheme we have the rate $3/4$ (33,4)-encoder combined with 16-level CPFSK modulation and $h=2/17$. This scheme has an asymptotic gain of ≈ 4.2 dB and the bandwidth is only $\approx 74\%$ of the MSK bandwidth. It is also seen in fig. 21 that schemes have been found which have the same asymptotic error performance as MSK but with roughly only half the MSK bandwidth. See also the examples given in fig. 17. Furthermore, schemes with the same spectral efficiency as the MSK scheme but with an asymptotic performance gain of 4-6 dB have been found.

The performance of some 4-level CPFSK 2-h multi-h schemes, [7]-[8], are also shown in fig. 21 (A-D, see table 9) for comparison. These are clearly outperformed by the coded CPFSK schemes considered in this part.

To illustrate how the power-bandwidth tradeoff is affected by the definition of bandwidth, fig. 22 is studied. This figure shows the same as fig. 21 does, but with the 99.9% in band power definition of bandwidth (the -30 dB level in the fractional out of band power function). By comparing figs. 21-22 it is seen that the principal characteristics are roughly the same in these two figures.

V. DISCUSSION AND CONCLUSIONS

Good coded modulation schemes have been found by utilizing a CPFSK modulation scheme with a large number of levels, a low modulation index and a convolutional code with high rate. From the results above we can conclude that we have given explicit coded CPFSK schemes with approximately half the bandwidth of minimum shift keying (MSK) and the same asymptotic error probability. Also, explicit schemes with about 4-6dB asymptotic gain in E_b/N_0 with roughly the same bandwidth as MSK are given. By varying the system parameters, mainly the modulation index, schemes with different combinations of bandwidth saving and power gain can be obtained.

Code-independent and code-specific upper bounds on the free Euclidean distance have been identified for an interesting subclass of coded multilevel CPM schemes with a natural binary mapper. These give good insight into the relationship between systems parameters and are of great value in the search for good codes. The code-independent upper bound on the free Euclidean distance is reached for certain parameter combinations. When this happens, there is no point in further increase the code memory for the given restricted class of codes. To obtain further possible increases, more general mappers and convolutional encoders have to be considered. How this selection should be made is not clear at this point.

A class of smooth coded partial response CPM schemes with short pulse memory is also considered. Significantly lower spectral sidelobes compared to CPFSK are obtained where the good Euclidean distance properties are almost preserved. The low sidelobes are achieved with a very modest degree of smoothing. The phase trajectories are very similar to those of CPFSK except the smooth phase changes. The best codes found for CPFSK are also used for this class of partial response schemes.

By introducing memory into the signals by means of continuous phase, a convolutional code and in some cases partial response (correlative encoding), schemes with constant amplitude signals, excellent power spectra, and good error probability properties can be obtained. Good detection properties require a receiver which makes sequence detection. Increasing signal memory leads to improved performance but also to increased optimum receiver complexity. From eq. (11) in Part I (with $\eta = L-1$ and $k'=k$) it is found that the number of states S in the Viterbi algorithm grows exponentially with v and $(L-1)k$. S also grows with p .

Figure 23 summarizes the principal results in this part and in Part II. $\square$ represents the best found rate $1/2$ encoders with binary CPFSK modulation and $\diamond$ represents the best found rate $1/2$ encoders with 4-level CPFSK modulation. It is seen in this figure that the schemes considered in this part is significantly more bandwidth efficient than the schemes considered in Part II.

We have only considered the asymptotic error probability properties of the coded systems (determined by the free Euclidean distance). There are good reasons for this. It is difficult enough to find good schemes in terms of the free Euclidean distance. Schemes with good free distance normally also are good at error probability levels which are relevant in the applications. This is the subject of Part IV where the bit error probability for coded CPM schemes are studied.

APPENDIX A: AN ALGORITHM TO CALCULATE THE MINIMUM EUCLIDEAN DISTANCE

This algorithm [14] is based on the state description of the transmitted signal $s(t, \underline{\alpha})$, see section II in Part I. Let us first define, at level j , the two-dimensional array A_j , with elements $A_j(\ell, m)$,

$$A_j(\ell, m) = \min_{\substack{\underline{u}, \underline{w} \\ \underline{u}_r^1 \neq \underline{w}_r^1 \\ \underline{u}_r^1 = \underline{w}_r^1 = -1}} \frac{1}{2E_b} \int_0^{jN^1 T} \left(s(t, \underline{\alpha}) - s(t, \underline{\beta}) \right)^2 dt \quad (A1)$$

where the minimization is over all pairs $(\underline{u}, \underline{w})$ such that one of the two signals, $s(t, \underline{\alpha})$ or $s(t, \underline{\beta})$, is in state ℓ in level j and the other signal is in state m in level j . Furthermore, the sequences $\underline{u}$ and $\underline{w}$ must also be such that $\underline{u}_r^1 = \underline{w}_r^1$ for all $r \leq -1$.

From the definition of the matrix A_j it is seen that

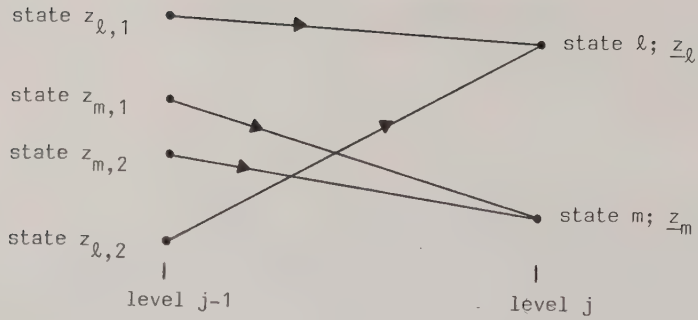
$$d_{\min, N}^2(h) = \min_{\ell, m} A_j(\ell, m), \quad N = jN^1 \quad (A2)$$

An upper bound on $d_{\min}^2(h)$, $d_B^2(h)$, can also be found

$$d_B^2(h) = A_j(\ell, \ell) \quad (A3)$$

since the diagonal elements in the matrix A_j is the normalized squared Euclidean distance associated with phase and state mergers.

Now assume that we have a list of all states which leads to state ℓ in one step, $\underline{z}_\ell = (z_{\ell, 1}, \dots, z_{\ell, s})$ where $s = 2^{k^1}$. This is illustrated below.



The algorithm can now be formulated as

$$A_j(l, m) = \min_{a, b} \left(A_{j-1}(z_{l,a}, z_{m,b}) + d^2(z_{l,a}, z_{m,b}) \right) \quad (A4)$$

where the range of a and b is $1 \leq a \leq 2^{k'}$ and $1 \leq b \leq 2^{k'}$. $d^2(z_{l,a}, z_{m,b})$ is the normalized squared Euclidean distance contribution over the $(j-1)$:th interval.

The algorithm stops at level j_* if

$$A_{j_*}(l, m) \geq d_B^2(h) = \min_{1 \leq j \leq j_*} \{ \min_i A_j(i, i) \} ; \text{ all } l, m \quad (A5)$$

When the algorithm stops it is clear that $d_{\min}^2(h) = d_B^2(h)$.

Rate 2/3 convolutional encoder and 8-level CPFSK modulation		
h	$d_{\min}^2(h)$	Optimal $v=1$ rate 1/2 conv. enc., N, N_m
1/11(0.0909)	0.8218	$\{(1,2),8,3\},\{(2,1),3,3\},\{(3,1),3,3\},\{(3,2),8,3\}$
2/21(0.0952)	0.8970	$\{ \quad \quad \quad \},\{ \quad \quad \quad \},\{ \quad \quad \quad \},\{ \quad \quad \quad \}$
1/10(0.1000)	0.9826	$\{ \quad \quad \quad \},\{ \quad \quad \quad \},\{ \quad \quad \quad \},\{ \quad \quad \quad \}$
2/19(0.1053)	1.0807	$\{ \quad \quad \quad \},\{ \quad \quad \quad \},\{ \quad \quad \quad \},\{ \quad \quad \quad \}$
1/9(0.1111)	1.1937	$\{ \quad \quad \quad \},\{ \quad \quad \quad \},\{ \quad \quad \quad \},\{ \quad \quad \quad \}$
2/17(0.1176)	1.3246	$\{ \quad \quad \quad \},\{ \quad \quad \quad \},\{ \quad \quad \quad \},\{ \quad \quad \quad \}$
1/8(0.1250)	1.4771	$\{ \quad \quad \quad \},\{ \quad \quad \quad \},\{ \quad \quad \quad \},\{ \quad \quad \quad \}$
2/15(0.1333)	1.6560	$\{ \quad \quad \quad \},\{ \quad \quad \quad \},\{ \quad \quad \quad \},\{ \quad \quad \quad \}$
1/7(0.1429)	1.8670	$\{ \quad \quad \quad \},\{ \quad \quad \quad \},\{ \quad \quad \quad \},\{ \quad \quad \quad \}$
2/13(0.1538)	2.1175	$\{ \quad \quad \quad \},\{ \quad \quad \quad \},\{ \quad \quad \quad \},\{ \quad \quad \quad \}$
1/6(0.1667)	2.4164	$\{ \quad \quad 7,3\},\{ \quad \quad \quad \},\{ \quad \quad \quad \},\{ \quad \quad 7,3\}$
2/11(0.1818)	2.7741	$\{ \quad \quad \quad \},\{ \quad \quad 10,3\},\{ \quad \quad 10,3\},\{ \quad \quad \quad \}$
1/5(0.2000)	3.2022	$\{ \quad \quad \quad \},\{ \quad \quad 4,3\}$
2/9(0.2222)	3.7109	$\{ \quad \quad 13,3\},\{ \quad \quad 5,3\},\{(3,1),5,3\},\{(3,2),13,3\}$
1/4(0.2500)	<u>2.0000</u>	$\{ \quad \quad 3,1\},\{ \quad \quad 1,1\},\{ \quad \quad 1,1\},\{ \quad \quad 3,1\}$

Table 1. The optimal $d_{\min}^2(h)$ and corresponding optimal encoders when a rate 2/3 convolutional encoder and 8-level CPFSK modulation are used, $v=1$. The $d_{\min}^2(h)$ value is underlined if the code-independent upper bound is reached.

Rate 2/3 conv. enc. and 8-level CPFSK modulation		
h	$d_{\min}^2(h)$	Optimal $v=2$ rate 1/2 conv. enc., N, N_m
1/11	1.4297	$\{(5,2), 12, 4\}, \{(7,2), 12, 4\}$
2/21	1.5595	$\{ \text{ " " " }, \{ \text{ " " " }\}$
1/10	1.7072	$\{ \text{ " " " }, \{ \text{ " " " }\}$
2/19	1.8761	$\{ \text{ " " " }, \{ \text{ " " " }\}$
1/9	2.0703	$\{ \text{ " " " }, \{ \text{ " " " }\}$
2/17	2.2947	$\{ \text{ " " " }, \{ \text{ " " " }\}$
1/8	2.5554	$\{ \text{ " " " }, \{ \text{ " " " }\}$
2/15	2.8602	$\{ \text{ " " " }, \{ \text{ " " " }\}$
1/7	3.2181	$\{ \text{ " " " }, \{ \text{ " " " }\}$
2/13	3.6406	$\{ \text{ " } 21, 4\}, \{ \text{ " } 21, 4\}$
1/6	4.1407	$\{ \text{ " } 11, 4\}, \{ \text{ " } 11, 4\}$
2/11	3.8994	$\{(3,4), 20, 5\}, \{(6,1), 19, 5\}, \{(7,1), 19, 5\}, \{(7,4), 17, 5\}$
1/5	4.4636	$\{(5,2), 10, 3\}, \{(7,2), 10, 3\}$
2/9	<u>4.4604</u>	$\{(1,4), 18, 2\}, \{(3,4), 18, 2\}, \{(4,1), 9, 2\}, \{(5,2), 9, 2\}$ $\{(6,1), 12, 2\}, \{(7,1), 12, 2\}, \{(7,2), 9, 2\}, \{(7,4), 18, 2\}$

Table 2. The optimal $d_{\min}^2(h)$ and corresponding optimal encoders when a rate 2/3 convolutional encoder and 8-level CPFSK modulation are used, $v=2$.

Rate 2/3 conv. enc. and 8-level CPFSK modulation		
h	$d_{\min}^2(h)$	Optimal $v=3$ rate 1/2 conv. enc., N, N_m
1/11	1.8372	$\{(13,4), 23, 5\}, \{(15,2), 18, 5\}, \{(17,2), 18, 5\}, \{(17,4), 23, 5\}$
2/21	2.0039	$\{ \quad \quad \quad \}, \{ \quad \quad \quad \}, \{ \quad \quad \quad \}, \{ \quad \quad \quad \}$
1/10	2.1935	$\{ \quad \quad \quad \}, \{ \quad \quad \quad \}, \{ \quad \quad \quad \}, \{ \quad \quad \quad \}$
2/19	2.4104	$\{ \quad \quad \quad \}, \{ \quad \quad \quad \}, \{ \quad \quad \quad \}, \{ \quad \quad \quad \}$
1/9	2.6596	$\{ \quad \quad \quad \}, \{ \quad \quad \quad \}, \{ \quad \quad \quad \}, \{ \quad \quad \quad \}$
2/17	2.9476	$\{ \quad \quad \quad \}, \{ \quad \quad \quad \}, \{ \quad \quad \quad \}, \{ \quad \quad \quad \}$
1/8	3.2822	$\{ \quad \quad \quad \}, \{ \quad \quad \quad \}, \{ \quad \quad \quad \}, \{ \quad \quad \quad \}$
2/15	3.6730	$\{ \quad \quad 22, 5\}, \{ \quad \quad \quad \}, \{ \quad \quad \quad \}, \{ \quad \quad 22, 5\}$
1/7	4.1320	$\{ \quad \quad \quad \}, \{ \quad \quad 16, 5\}, \{ \quad \quad 16, 5\}, \{ \quad \quad \quad \}$
2/13	4.6733	$\{ \quad \quad 34, 5\}, \{ \quad \quad 32, 5\}, \{ \quad \quad 32, 5\}, \{ \quad \quad 34, 5\}$
1/6	4.7624	$\{(11,2), 14, 5\}, \{(11,4), 15, 5\}, \{(13,2), 14, 5\}, \{(15,4), 15, 5\}$
2/11	4.8170	$\{(3,10), 26, 5\}, \{(14,1), 25, 5\}$
1/5	<u>4.7568</u>	$\{(11,2), 11, 2\}, \{(11,4), 13, 2\}, \{(13,2), 11, 2\}, \{(13,4), 13, 2\}$ $\{(15,2), 14, 2\}, \{(15,4), 13, 2\}, \{(17,2), 14, 2\}, \{(17,4), 13, 2\}$

Table 3. The optimal $d_{\min}^2(h)$ and corresponding optimal encoders when a rate 2/3 convolutional encoder and 8-level CPFSK modulation are used, $v=3$.

Rate 2/3 convolutional encoder and 8-level CPFSK modulation		
h	$d_{\min}^2(h)$	Optimal v=4 rate 1/2 convolutional encoder, N, N_m
1/11	2.1615	$\{(33,4), 28, 7\}, \{(37,4), 28, 7\}$
2/21*	2.3596	{ " " " }
1/10*	2.5854	{ " " " }
2/19*	2.8442	{ " " " }
1/9*	3.1425	{ " " " }
2/17*	3.4882	{ " " " }
1/8*	3.8916	{ " " " }
2/15*	4.2482	{ " 26, 2 }
1/7*	4.4834	{ " 23, 2 }
2/13*	4.6860	{ " 32, 2 }
1/6	4.8270	$\{(21,4), 17, 2\}, \{(21,16), 19, 2\}, \{(23,4), 18, 2\}, \{(23,10), 16, 2\}$ $\{(23,12), 16, 2\}, \{(23,16), 17, 2\}, \{(25,2), 17, 2\}, \{(25,4), 17, 2\}$ $\{(25,10), 15, 2\}, \{(27,2), 17, 2\}, \{(27,4), 18, 2\}, \{(27,16), 17, 2\}$ $\{(31,2), 18, 2\}, \{(31,4), 18, 2\}, \{(31,12), 16, 2\}, \{(31,16), 17, 2\}$ $\{(33,2), 18, 2\}, \{(33,10), 16, 2\}, \{(35,4), 18, 2\}, \{(35,10), 15, 2\}$ $\{(35,16), 17, 2\}, \{(37,16), 19, 2\}$
2/11	4.8664	$\{(7,20), 27, 2\}, \{(13,20), 29, 2\}, \{(17,34), 29, 2\}, \{(23,4), 26, 2\}$ $\{(23,34), 27, 2\}, \{(27,4), 26, 2\}, \{(27,20), 29, 2\}, \{(31,4), 28, 2\}$ $\{(31,7), 26, 2\}, \{(32,1), 18, 2\}, \{(33,1), 18, 2\}, \{(33,20), 27, 2\}$ $\{(34,1), 18, 2\}, \{(35,1), 18, 2\}, \{(35,4), 28, 2\}, \{(36,7), 26, 2\}$

Table 4. The optimal $d_{\min}^2(h)$ and corresponding optimal encoders when a rate 2/3 convolutional encoder and 8-level CPFSK modulation are used, v=4.

*) Best found results, no complete search.

Rate 3/4 conv. enc. and 16-level CPFSK modulation				
h	$v=1$		$v=2$	
	$d_{\min}^2(h)$	Enc., N, N_m	$d_{\min}^2(h)$	Enc., N, N_m
1/11	1.2327	{(2,1),3,3}	2.1445	{(5,2),12,4}
2/21	1.3455	{ " " " }	2.3392	{ " " " }
1/10	1.4739	{ " " " }	2.5608	{ " " " }
2/19	1.6210	{ " " " }	2.8141	{ " " " }
1/9	1.7905	{ " " " }	3.1054	{ " " " }
2/17	1.9868	{ " " " }	3.4420	{ " " " }
1/8	2.2156	{ " " " }	<u>3.0000</u>	{ " 8,1}
2/15	2.4839	{ " " " }	4.2903	{ " 12,4}
1/7	2.8005	{ " " " }	4.8272	{ " " " }
2/13	3.1763	{ " " " }	<u>5.2298</u>	{ " 19,2}
1/6	<u>3.0000</u>	{ " 2,1}	---	-----
2/11	3.8501	{ " 8,2}	<u>5.6019</u>	{(5,2),19,2}
1/5	3.3511	{ " 2,2}	<u>5.7661</u>	{ " 8,2}
2/9	3.6725	{ " 5,2}	6.0405	{ " 15,3}
1/4	<u>3.0000</u>	{ " 1,1}	---	-----

Table 5. The best found $d_{\min}^2(h)$ and corresponding encoders when a rate 3/4 convolutional encoder and 16-level CPFSK modulation are used, $v=1$ and $v=2$.

Rate 3/4 conv. enc. and 16-level CPFSK modulation				
h	$v=3$		$v=4$	
	$d_{\min}^2(h)$	Enc., N, N_m	$d_{\min}^2(h)$	Enc. N, N_m
1/11	2.7558	{(15,2),18,5}	3.2422	{(33,4),28,7}
2/21	3.0058	{ " " " }	3.5394	{ " " " }
1/10	3.2903	{ " " " }	3.8781	{ " " " }
2/19	3.6156	{ " " " }	4.2663	{ " " " }
1/9	3.9894	{ " " " }	4.7137	{ " " " }
2/17	4.4214	{ " " " }	5.2324	{ " " " }
2/15	5.5096	{ " 19,5}	<u>5.6359</u>	{ " 21,2}
1/7	<u>5.3467</u>	{ " 15,2}	---	-----
2/9	<u>6.3101</u>	{ " 14,2}	---	-----

Table 6. The best found $d_{\min}^2(h)$ and corresponding encoders when a rate 3/4 convolutional encoder and 16-level CPFSK modulation are used, $v=3$ and $v=4$.

Rate 4/5 conv. enc. and 32-level CPFSK modulation				
h	$v=1$		$v=2$	
	$d_{\min}^2(h)$	Enc., N, N_m	$d_{\min}^2(h)$	Enc., N, N_m
1/11	1.6437	{{(2,1),3,3}}	2.8593	{{(5,2),12,4}}
2/21	1.7939	{ " " " }	3.1189	{ " " " }
1/10	1.9652	{ " " " }	3.4143	{ " " " }
2/19	2.1614	{ " " " }	3.7522	{ " " " }
1/9	2.3874	{ " " " }	4.1405	{ " " " }
2/17	2.6491	{ " " " }	4.5893	{ " " " }
1/8	2.9542	{ " " " }	<u>4.0000</u>	{ " 8,1}
2/15	3.3119	{ " " " }	5.7203	{ " 16,4}
1/7	3.7340	{ " " " }	<u>4.0000</u>	{ " 5,1}
2/13	4.2350	{ " " " }	<u>6.9730</u>	{ " 19,2}
1/6	<u>4.0000</u>	{ " 2,1}	---	-----
2/11	5.1335	{ " 8,2}	<u>7.4692</u>	{{(5,2),19,2}}
1/5	<u>4.0000</u>	{ " 2,1}	---	-----
2/9	4.8966	{ " 5,2}	<u>7.7933</u>	{{(5,2),14,2}}
1/4	<u>4.0000</u>	{ " 1,1}	---	-----

Table 7. The best found $d_{\min}^2(h)$ and corresponding encoders when a rate 4/5 convolutional encoder and 32-level CPFSK modulation are used, $v=1$ and $v=2$.

Rate 4/5 conv. enc. and 32-level CPFSK modulation				
h	$v=3$		$v=4$	
	$d_{\min}^2(h)$	Enc., N, N_m	$d_{\min}^2(h)$	Enc., N, N_m
1/11	3.6744	{{(15,2),18,5}}	4.3230	{{(33,4),28,7}}
2/21	4.0077	{ " " " }	---	-----
1/10	<u>4.0000</u>	{ " 15,1}	---	-----
2/19	4.8208	{ " 18,5}	---	-----
1/9	5.3192	{ " " " }	6.2850	{{(33,4),28,7}}
2/17	5.8952	{ " 21,5}	---	-----
2/15	7.3119	{ " 27,5}	<u>7.5145</u>	{{(33,4),29,2}}

Table 8. The best found $d_{\min}^2(h)$ and corresponding encoders when a rate 4/5 convolutional encoder and 32-level CPFSK modulation are used, $v=3$ and $v=4$.

Scheme	h_1	h_2	$d_{\min}^2$	$2BT_{99\%}^b$
A	0.40	0.46	4.80	1.13
B	0.25	0.29	2.80	0.84
C	0.25	0.23	2.00	0.73
D	4/16	3/16	1.60	0.67

Table 9. Parameters of 4-level 2-h multi-h schemes from [7]-[8].

$2BT_b$ is the 99% bandwidth.

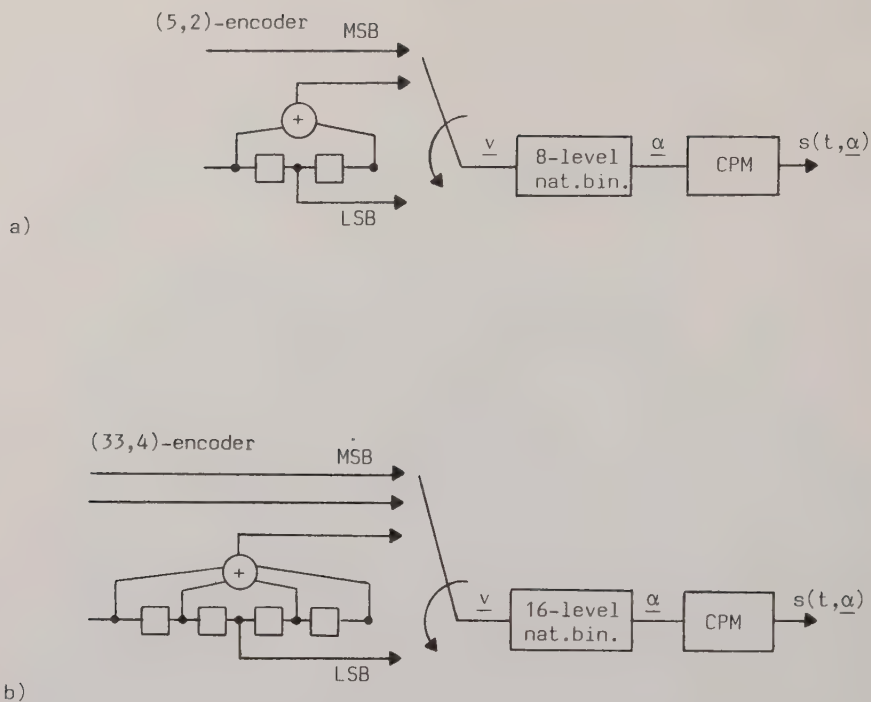


Figure 1. Examples on coded modulation schemes considered in this part.

- a) The rate $2/3$ $(5,2)$ -encoder combined with 8-level (nat.bin.) CPM.
- b) The rate $3/4$ $(33,4)$ -encoder combined with 16-level (nat.bin.) CPM.

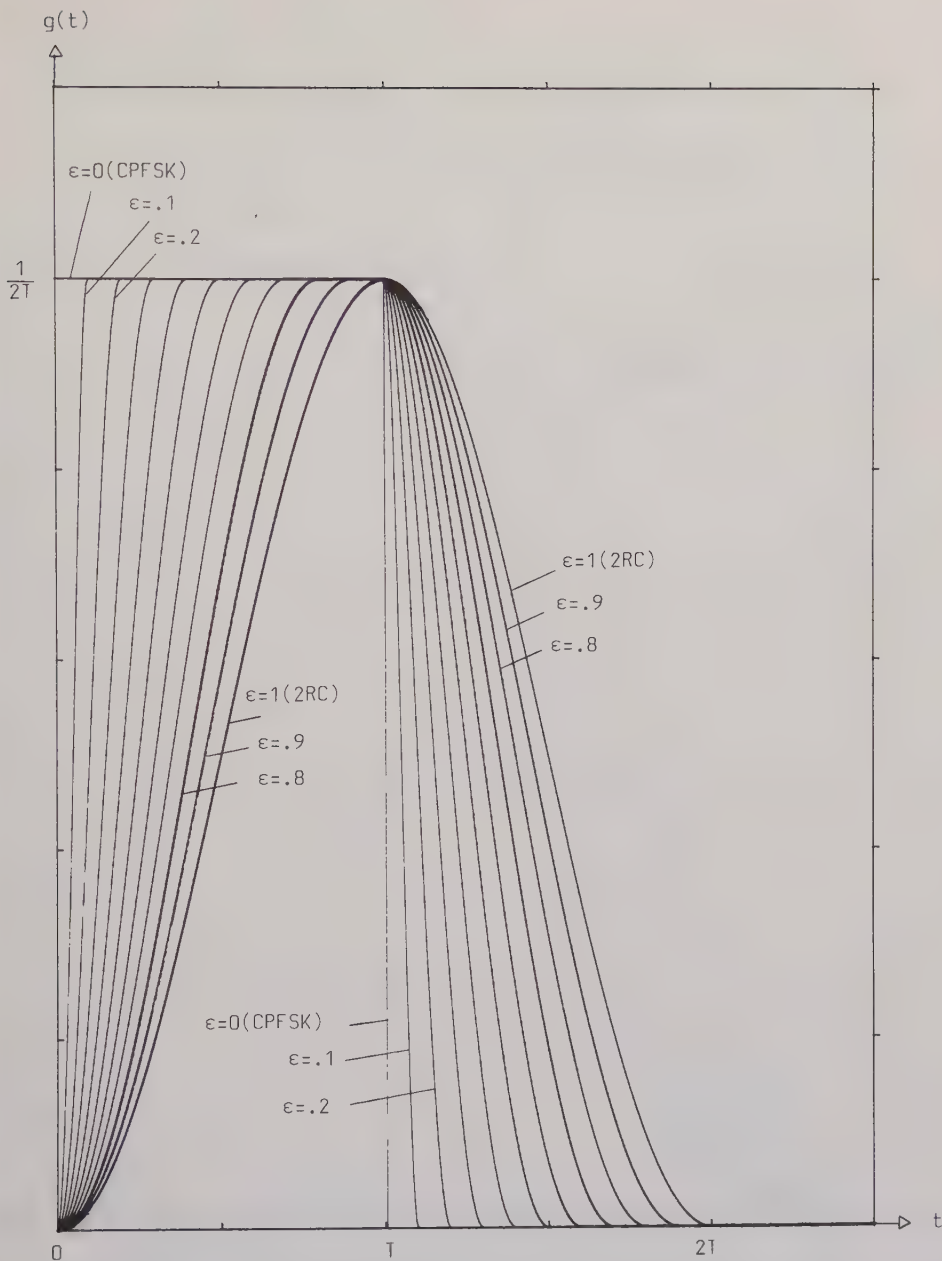


Figure 2. The frequency pulse $g(t)$ used in this part, see eq. (3). $\epsilon=0$ and $\epsilon=1$ corresponds to the CPFSK scheme and the 2RC scheme respectively.

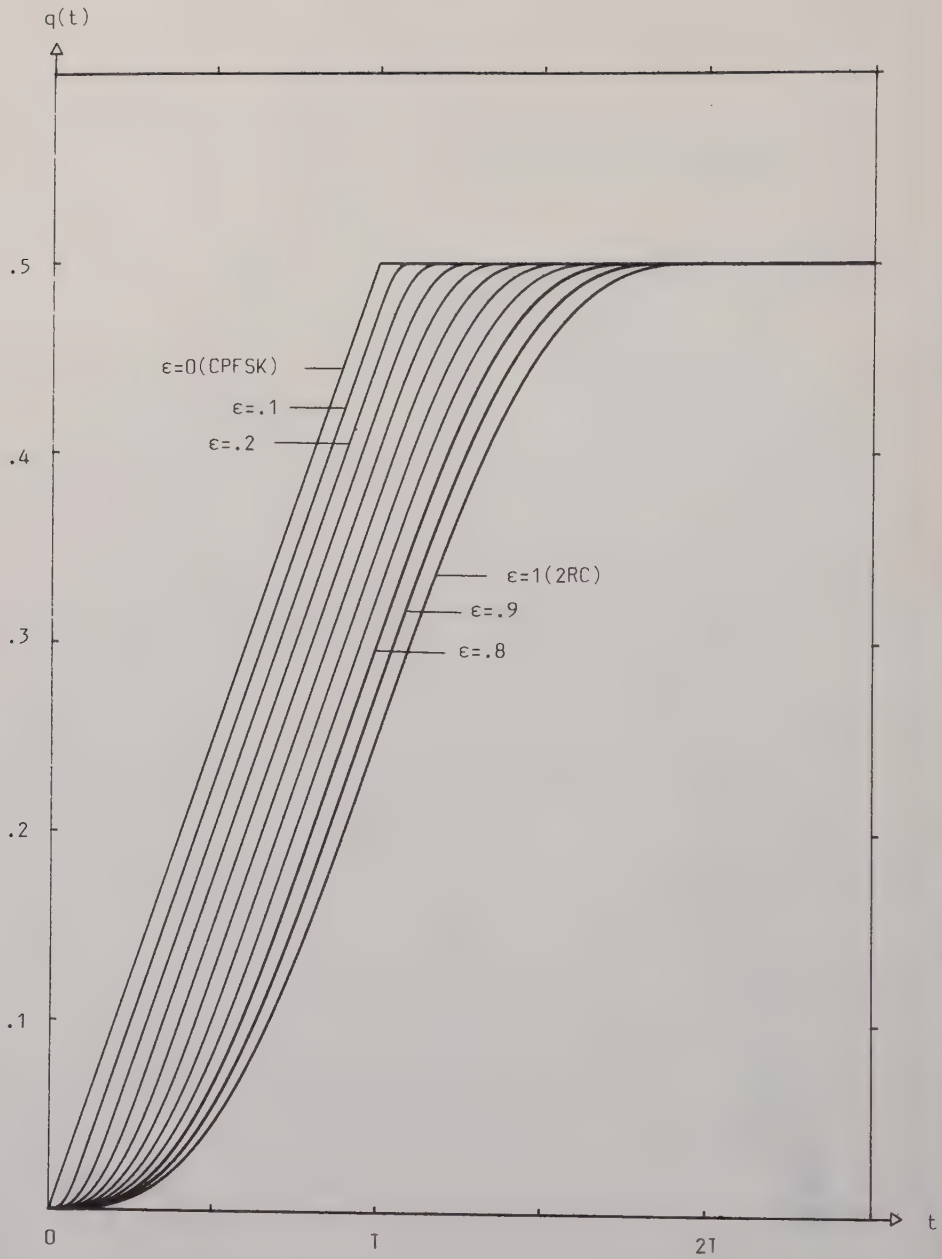


Figure 3. The phase response $q(t)$ used in this part, see eq. (A14) in Part I. $\epsilon=0$ and $\epsilon=1$ corresponds to the CPFSK scheme and the 2RC scheme respectively.

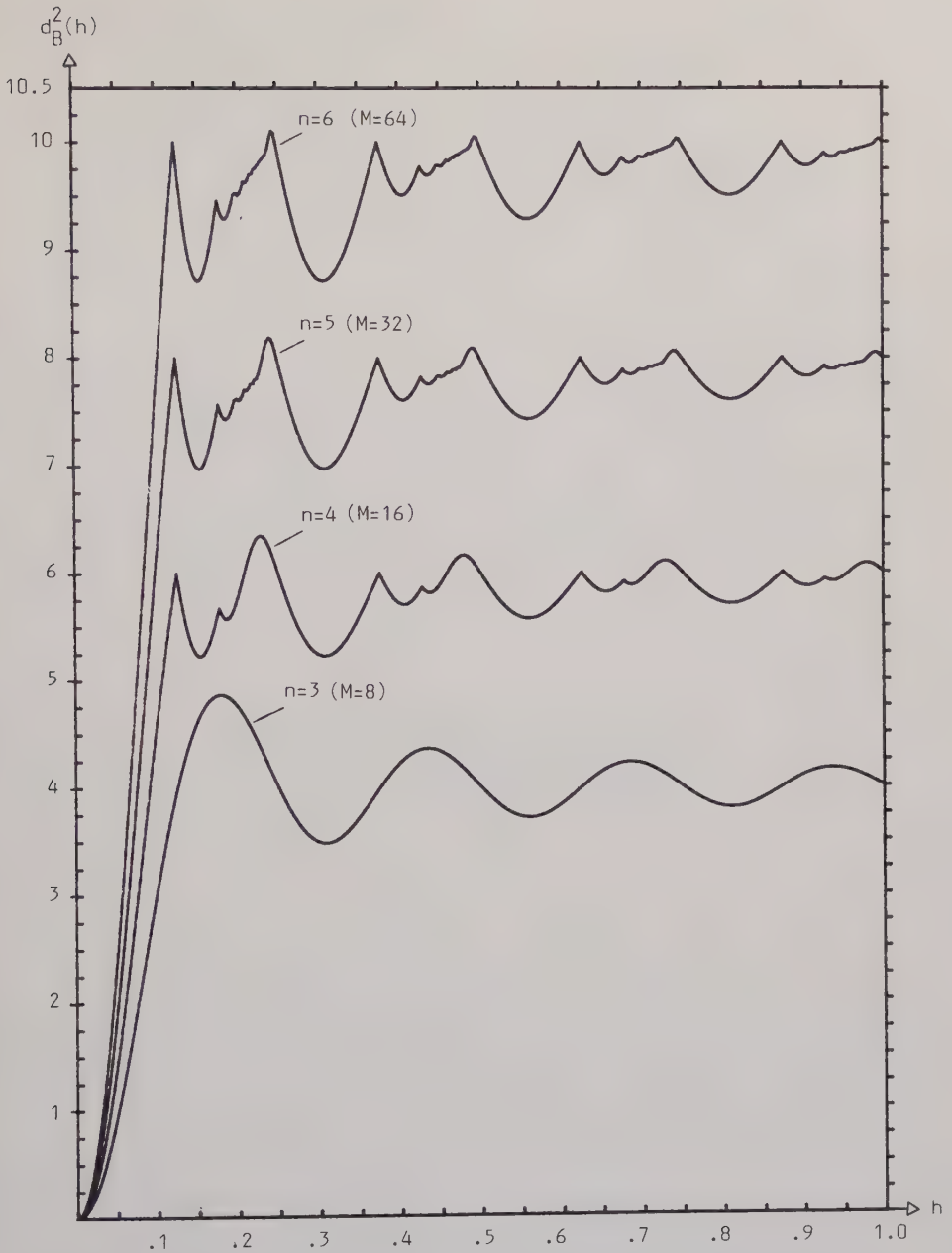
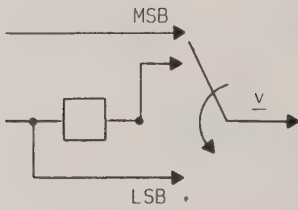
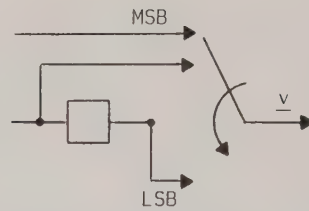


Figure 4. Upper bounds on $d_{\min}^2(h)$ for the sub-class of rate $(n-1)/n$ convolutional encoders and 2^n -level CPFSK modulation studied in this part. $n = 3, 4, 5$ and 6. See equation (5).

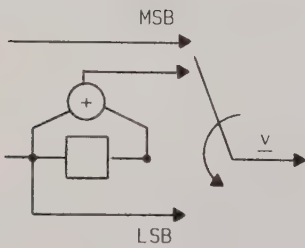
(1,2)-encoder



(2,1)-encoder



(3,2)-encoder



(3,1)-encoder

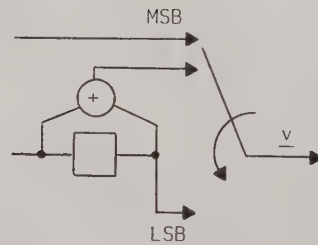


Figure 5. The optimal $v=1$ rate $2/3$ encoders when combined with 8-level CPFSK modulation with small modulation index h . See also table 1.

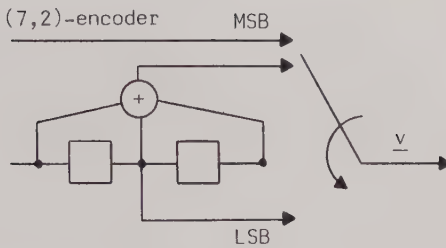
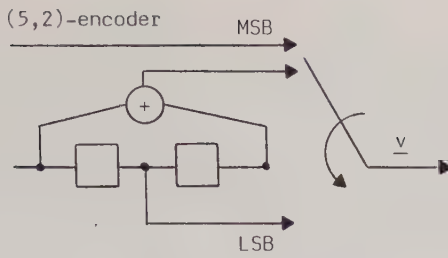


Figure 6. The optimal $v=2$ rate $2/3$ encoders when combined with 8-level CPFSK modulation with small modulation index h .
See also table 2.

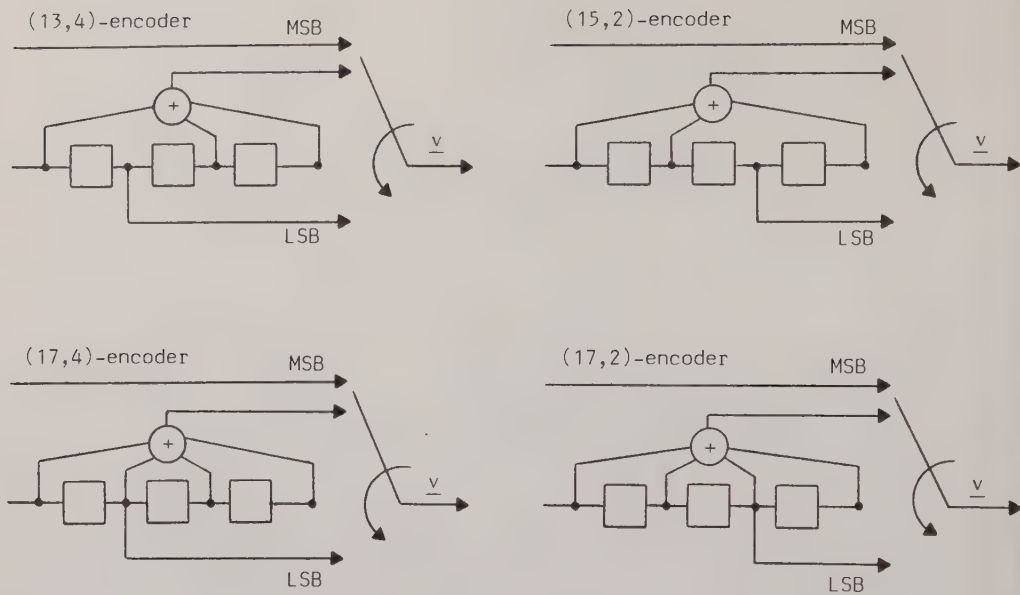


Figure 7. The optimal $v=3$ rate $2/3$ encoders when combined with 8-level CPFSK modulation with small modulation index h . See also table 3.

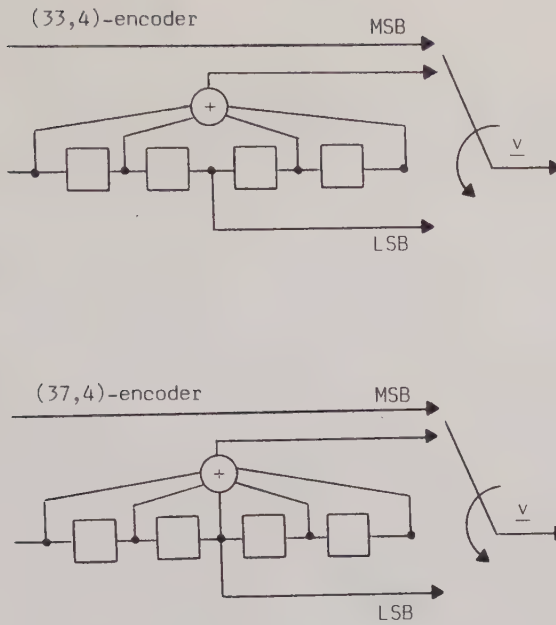


Figure 8. Good $v=4$ rate $2/3$ encoders when combined with 8-level CPFSK modulation with small modulation index h . See also table 4.

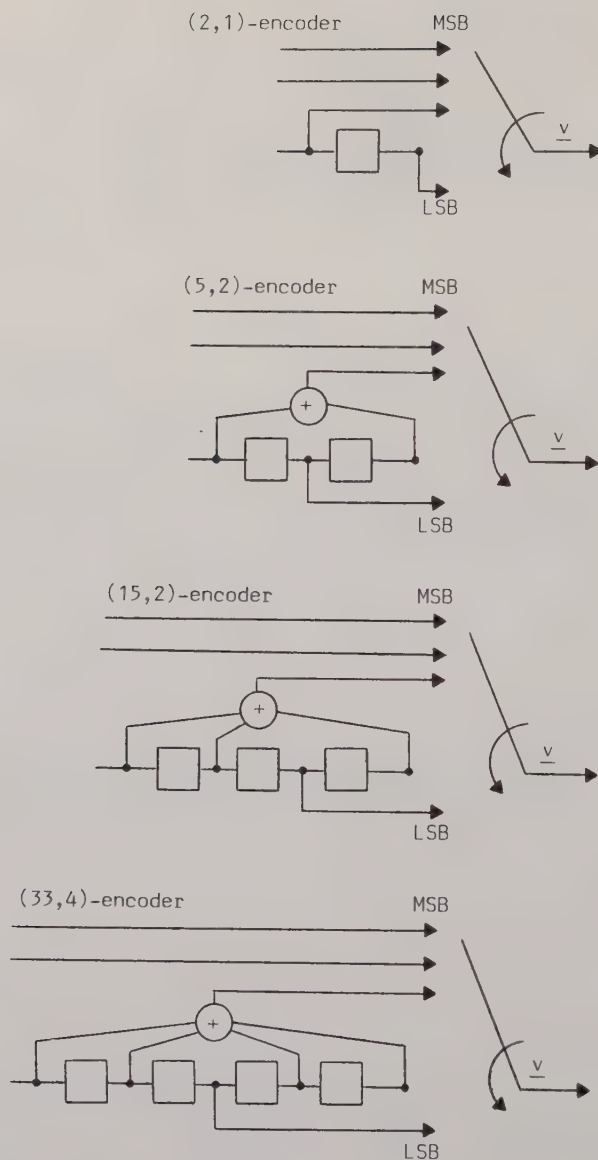
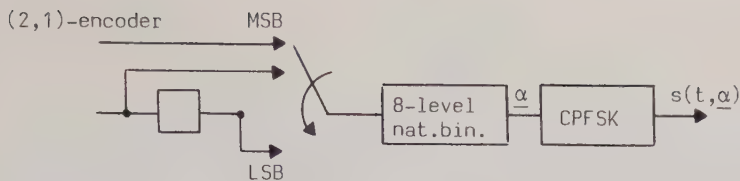


Figure 9. Examples of good rate 3/4 convolutional encoders when combined with 16-level CPFSK modulation (assuming small modulation indices). See also tables 5-6.



$$\underline{x} = (0)$$

$$\underline{u} = (01, 10, 00) \Rightarrow \underline{\alpha} = (-3, 3, -7)$$

$$\underline{w} = (10, 01, 00) \Rightarrow \underline{\beta} = (1, -3, -5) \Rightarrow \underline{\gamma} = (-4, 6, -2)$$

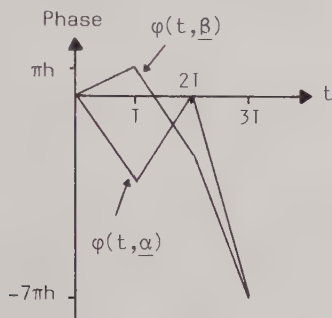
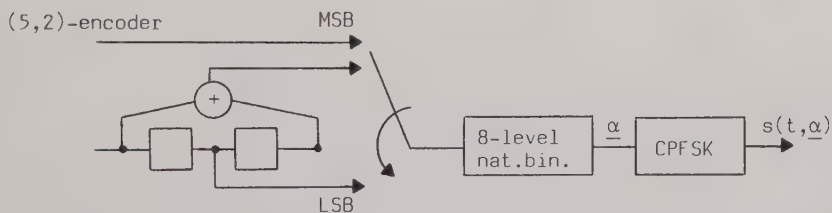


Figure 10. A minimum distance pair when the modulation index is small.



$$\underline{x} = (0, 0)$$

$$\underline{u} = (01, 10, 00, 01) \Rightarrow \underline{\alpha} = (-3, 3, -3, -3)$$

$$\underline{w} = (10, 01, 10, 01) \Rightarrow \underline{\beta} = (1, -3, 3, -7) \Rightarrow \underline{\gamma} = (-4, 6, -6, 4)$$

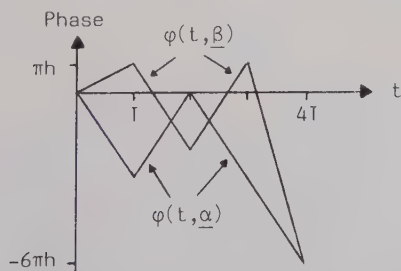


Figure 11. A minimum distance pair when the modulation index is small.

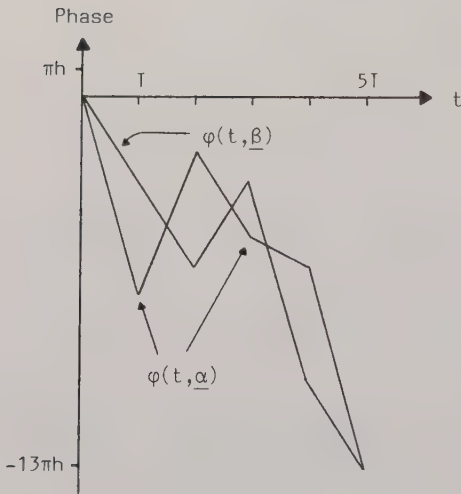
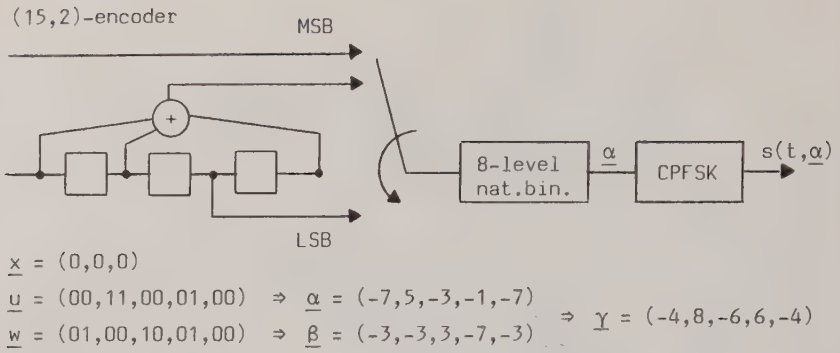
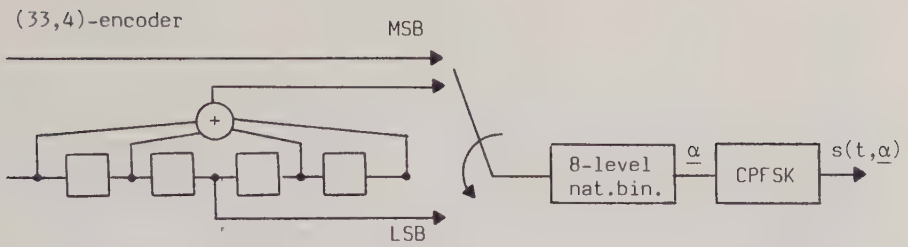


Figure 12. A minimum distance pair when the modulation index is small.



$$\underline{x} = (0, 0, 0, 0)$$

$$\underline{u} = (00, 01, 11, 00, 00, 00, 00) \Rightarrow \underline{\alpha} = (-7, -3, 1, -1, -1, -7, -3)$$

$$\underline{w} = (01, 01, 00, 00, 10, 00, 00) \Rightarrow \underline{\beta} = (-3, -7, -1, -1, 1, -3, -7) \Rightarrow \underline{\gamma} = (-4, 4, 2, 0, -2, -4, 4)$$

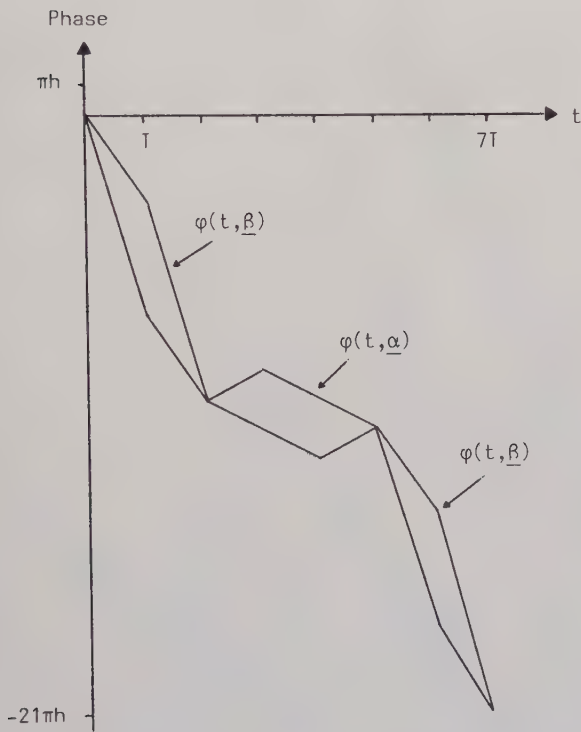


Figure 13. A minimum distance pair when the modulation index is small.

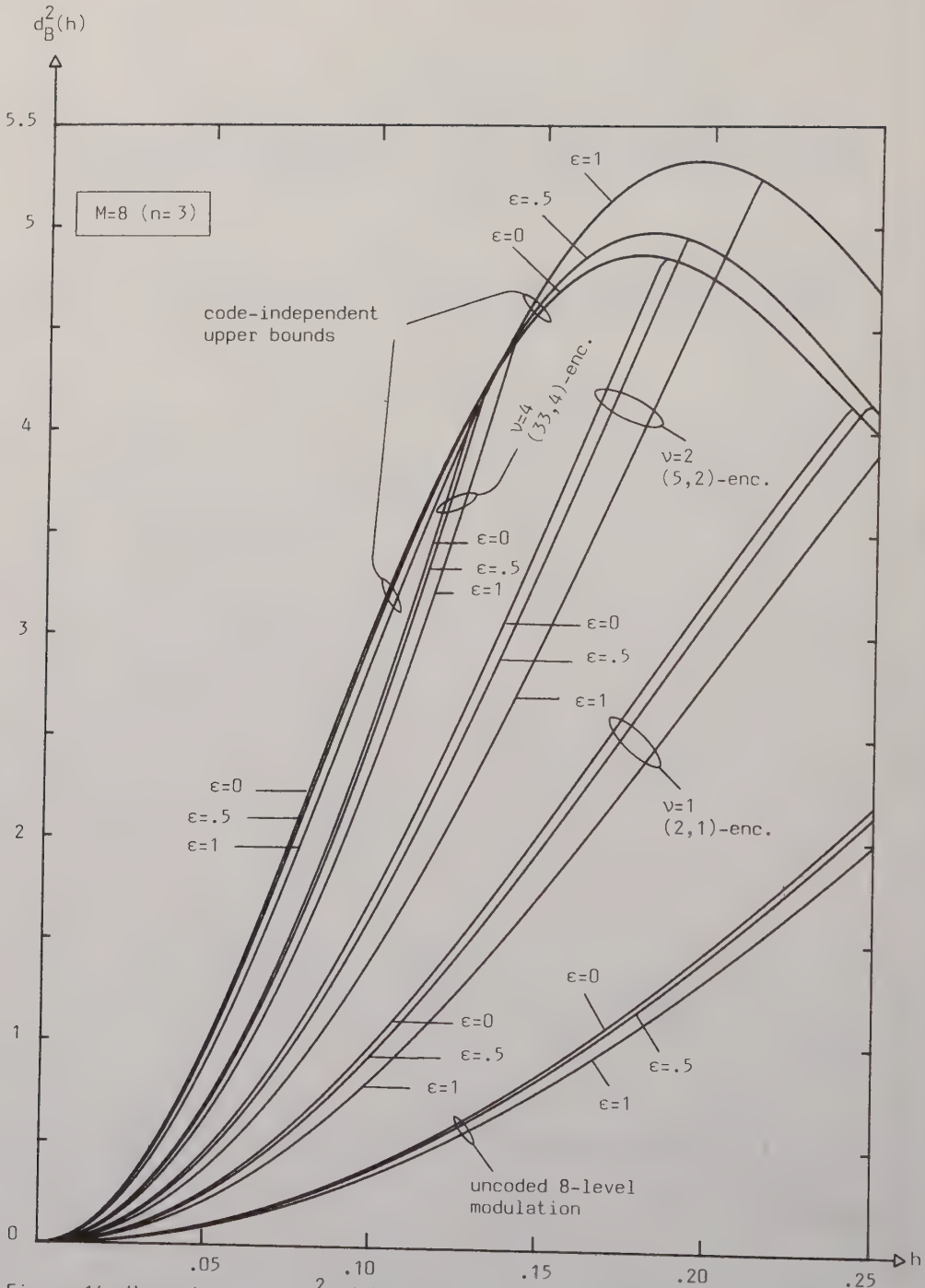


Figure 14. Upper bounds on $d_{\min}^2(h)$ when a rate $2/3$ convolutional encoder and 8-level CPM are used. The frequency pulse $g(t)$ follows equation (3).

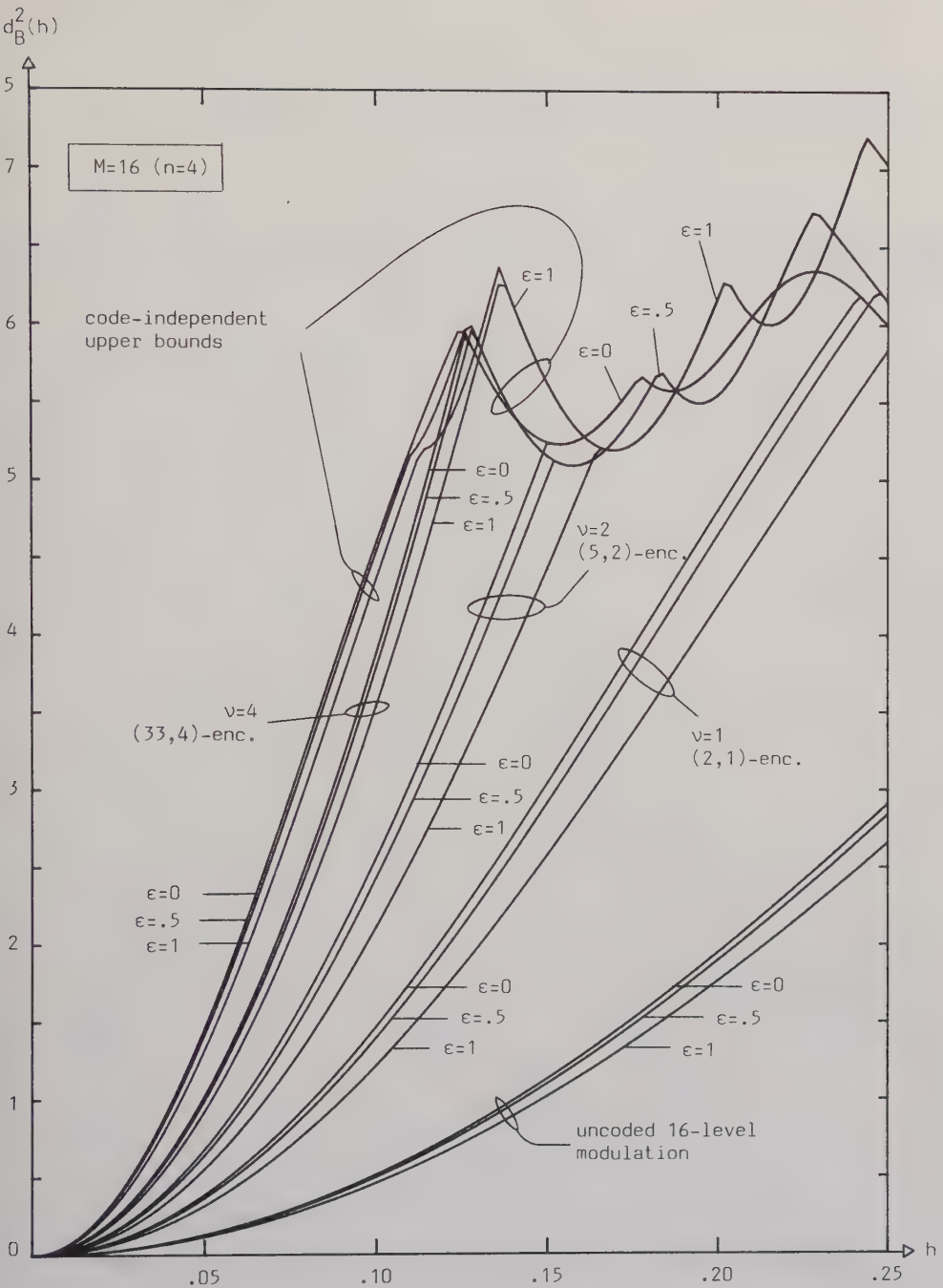


Figure 15. Upper bounds on $d_{\min}^2(h)$ when a rate $3/4$ convolutional encoder and 16-level CPM are used. The frequency pulse $g(t)$ follows equation (3).

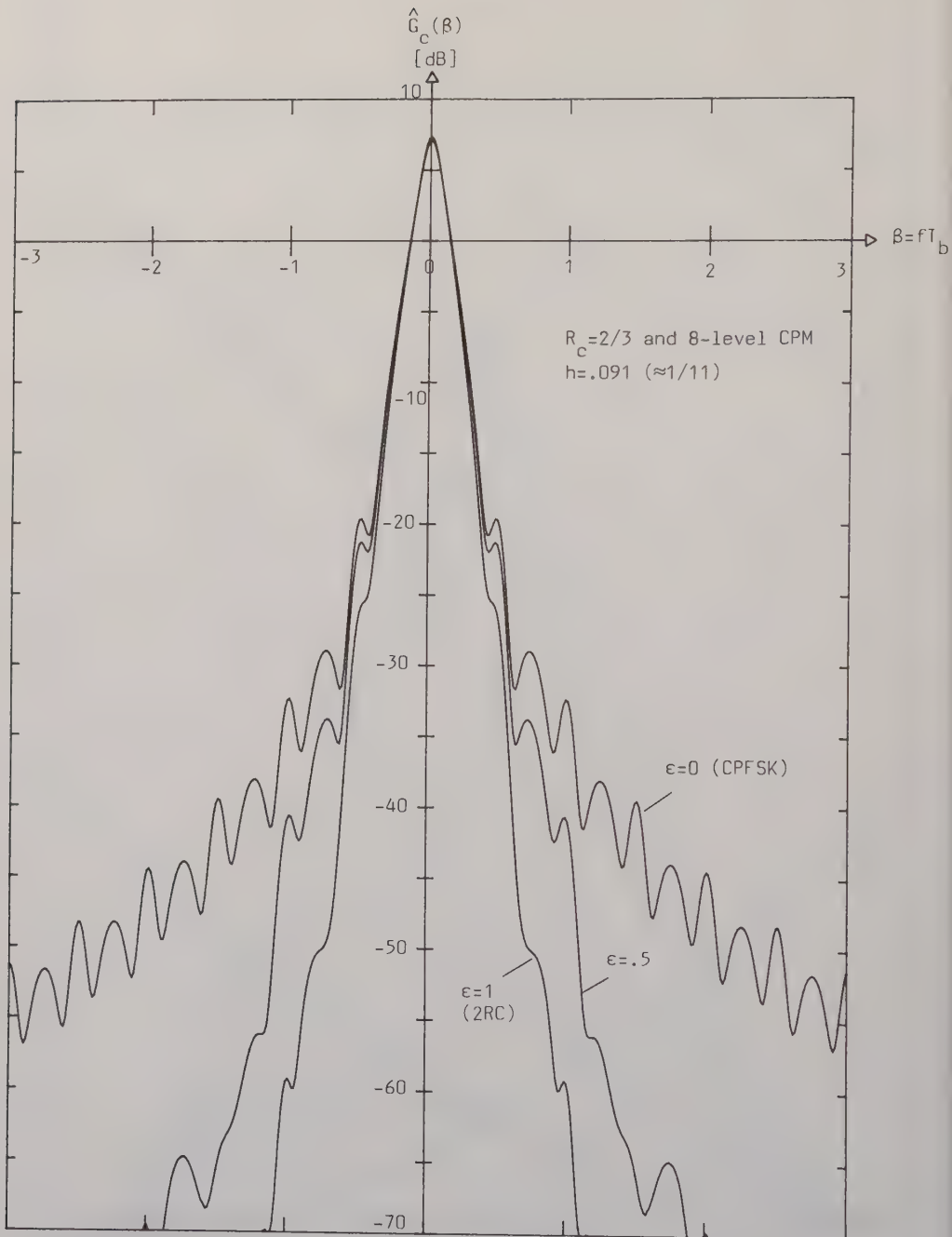


Figure 16. Examples of estimated coded power spectra ($\hat{G}_c(\beta)$) in decibel for different ϵ -values. The frequency pulse $g(t)$ is given in eq. (3).

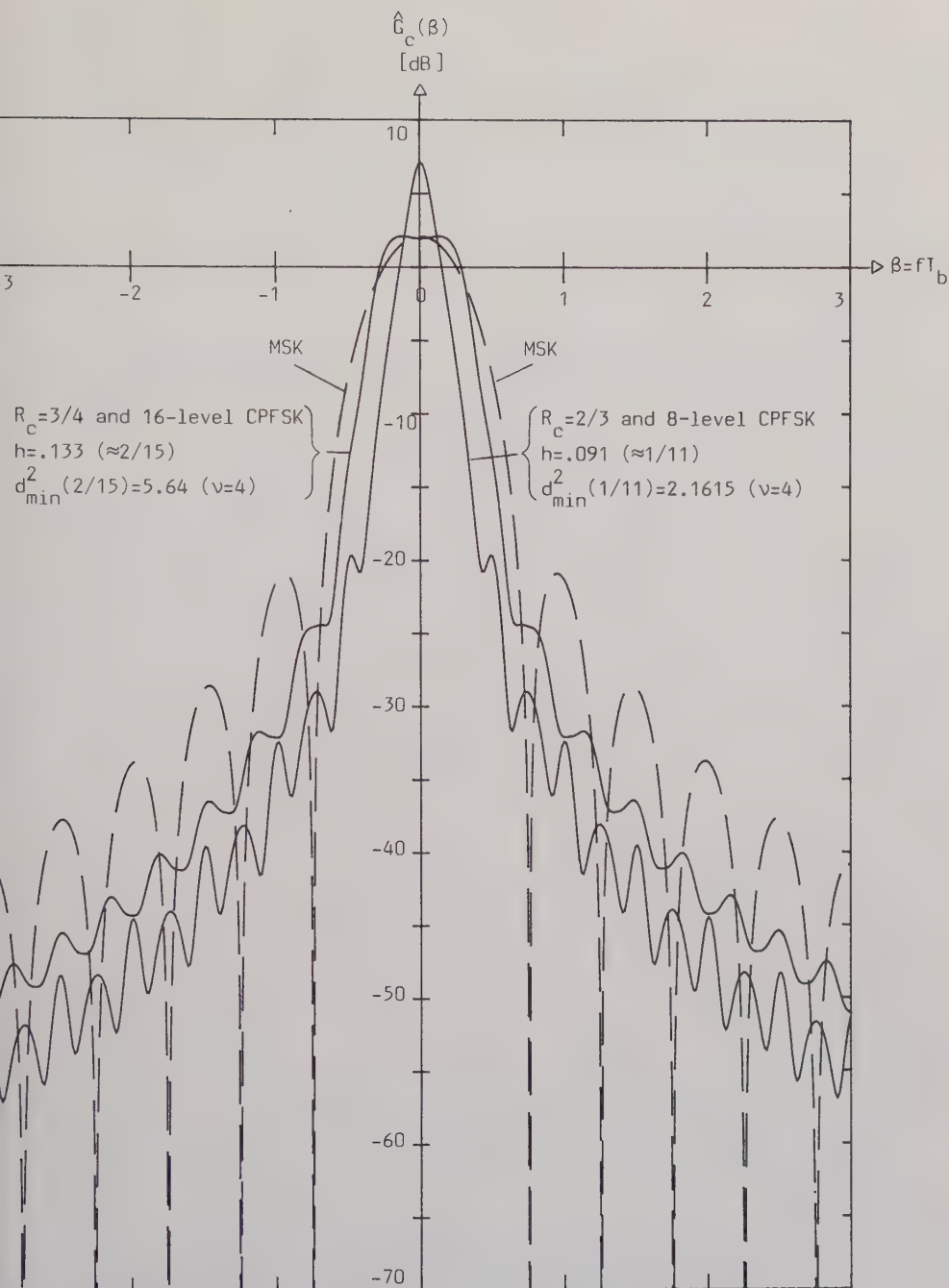


Figure 17. Estimated power spectra for two coded CPFSK schemes. The power spectra for uncoded MSK is given as a reference.

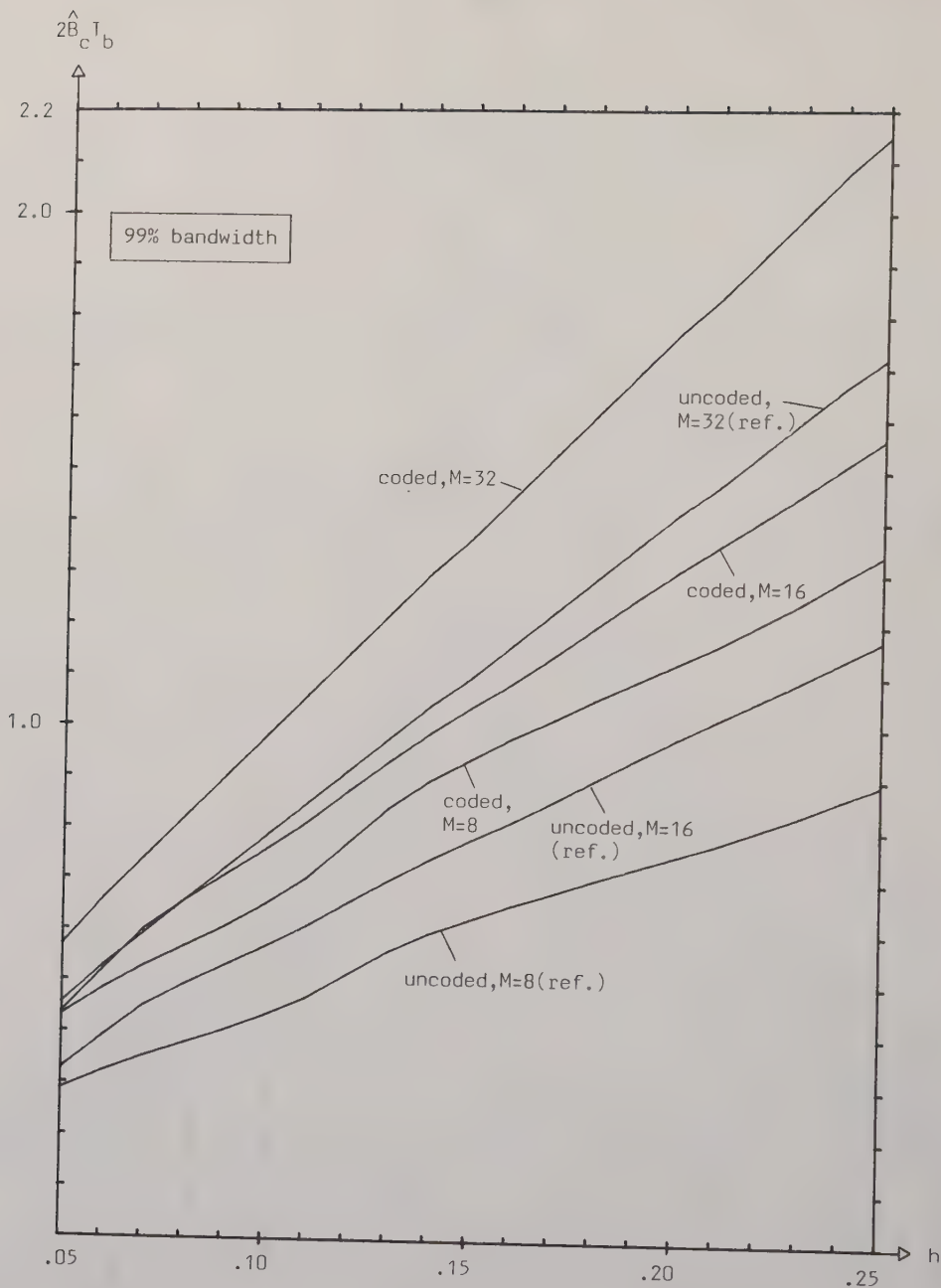


Figure 18. The estimated normalized double-sided bandwidth $2\hat{B}_c T_b (=2BT_b/R_c)$ as a function of the modulation index h when a rate $(n-1)/n$ convolutional encoder and 2^n -level CPFSK modulation are used, $n = 3, 4$ and 5 . The level in the fractional out of band power function used in the definition of bandwidth is $-20[\text{dB}]$. See also table 2 in Part I.

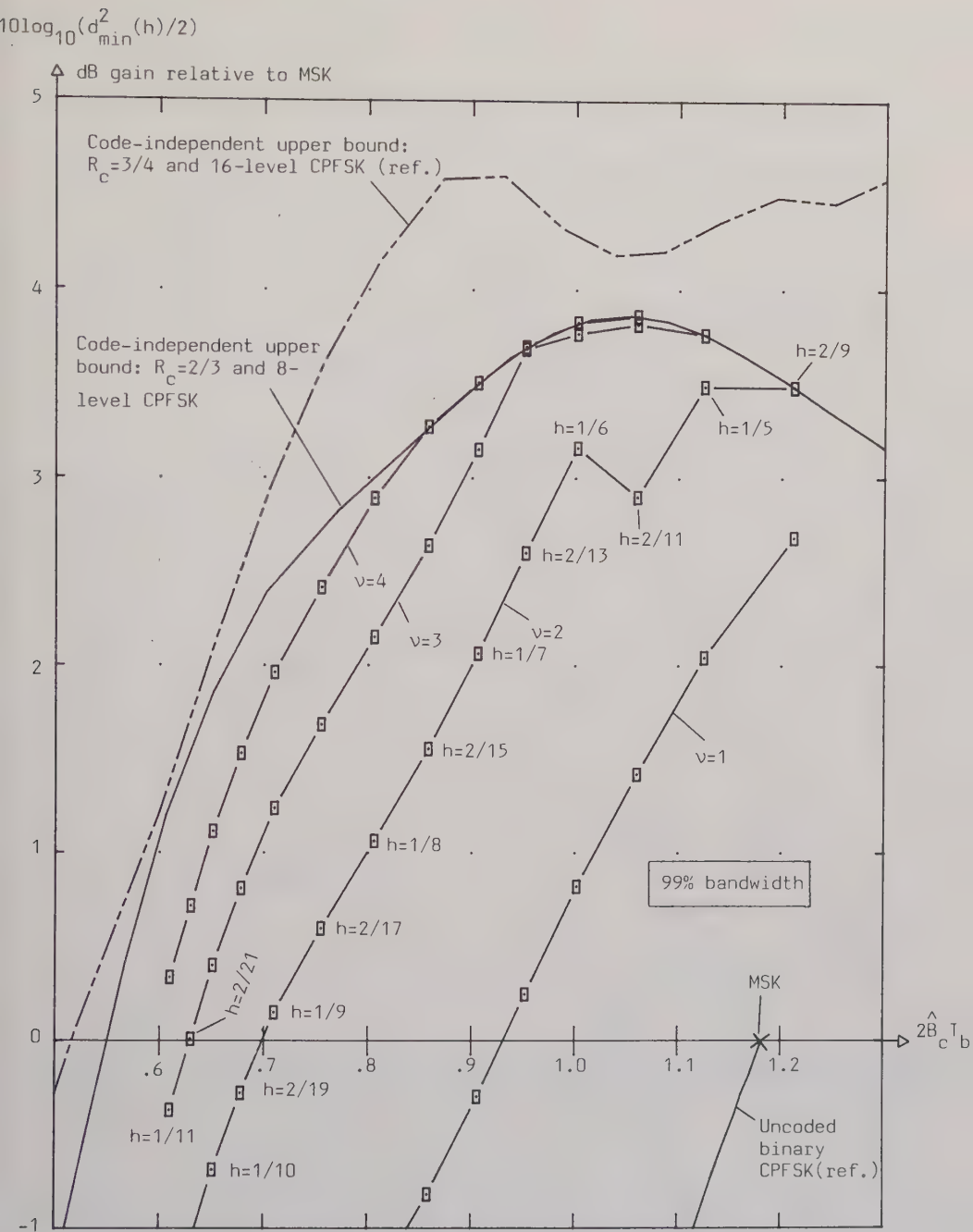


Figure 19. Power-bandwidth tradeoff for some coded 8-level CPFSK schemes. $v = 1, 2, 3$ and 4. The 99% in band power definition of bandwidth is used.

$$10 \log_{10} (d_{\min}^2(h)/2)$$

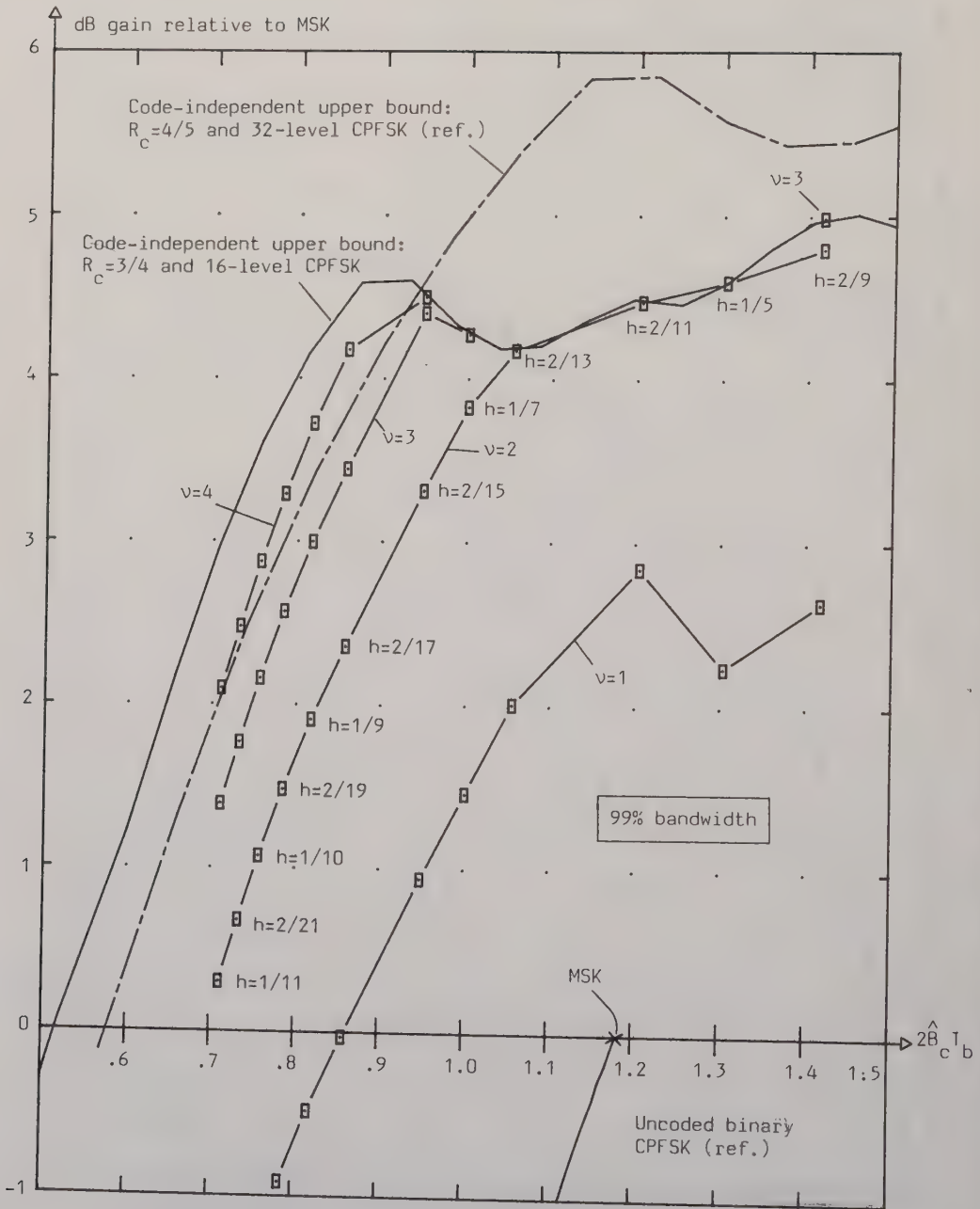


Figure 20. Power-bandwidth tradeoff for some coded 16-level CPFSK schemes. $v = 1, 2, 3$ and 4. The 99% in band power definition of bandwidth is used.

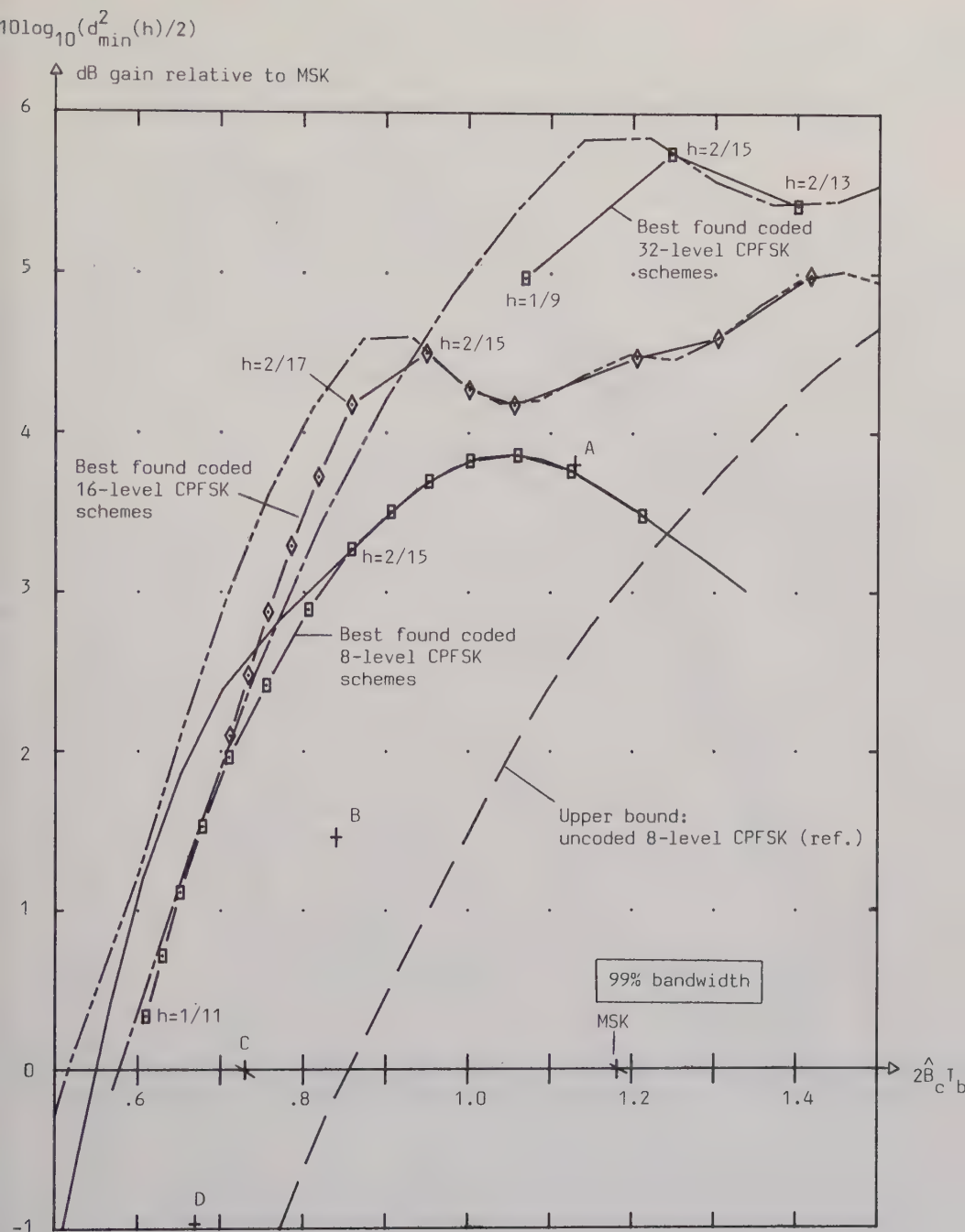


Figure 21. Power-bandwidth tradeoff for the best schemes found. As reference points some 4-level multi- h schemes have been given (A,B,C and D, see table 9). The 99% in band power definition of bandwidth is used.

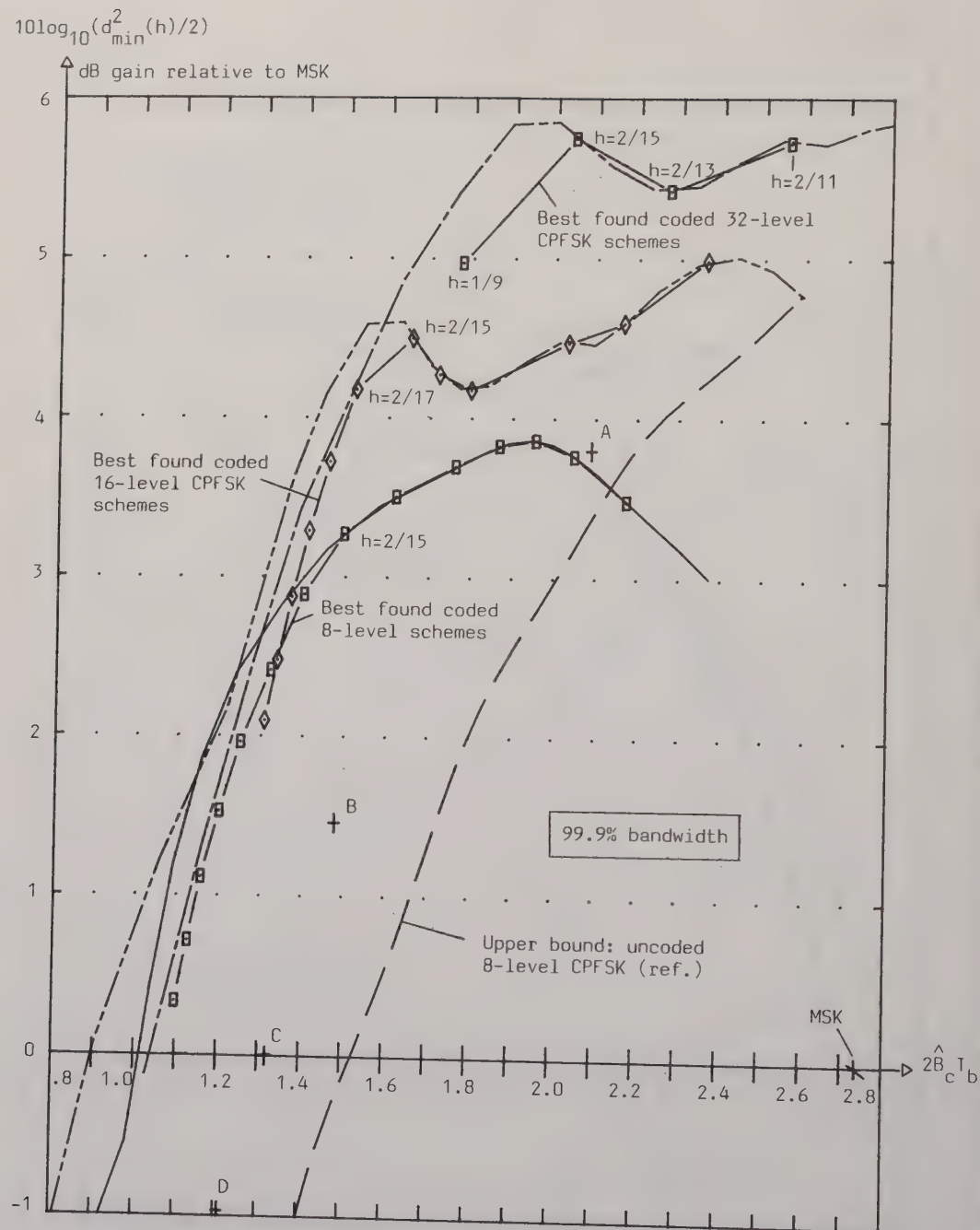


Figure 22. Same as figure 21 but with the 99.9% in band power definition of bandwidth.

$$10 \log_{10}(d_{\min}^2(h)/2)$$

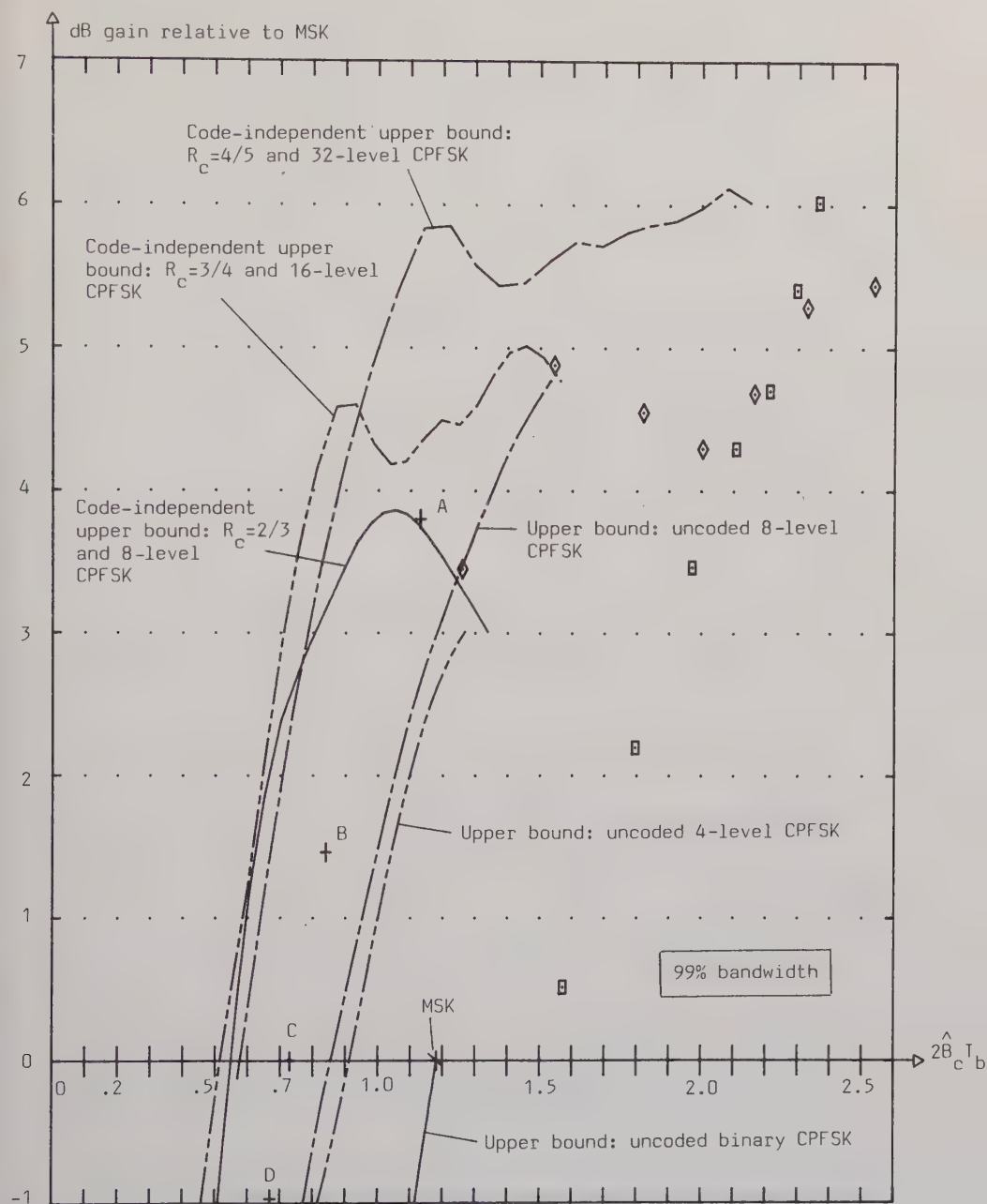


Figure 23. Power-bandwidth tradeoff. $\square$ represents the best found $R_c = 1/2$ and binary CPFSK schemes, see Part II. $\diamond$ represents the best found $R_c = 1/2$ and 4-level CPFSK schemes, see Part II.

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ON CODED
CONTINUOUS PHASE MODULATION

PART IV

AN UPPER BOUND ON THE BIT ERROR PROBABILITY

ABSTRACT

Continuous phase modulation (CPM) is a class of constant amplitude modulations containing schemes with good combined power and bandwidth efficiency.

In this part we study the bit error probability properties for modulation schemes consisting of a convolutional encoder and multi-level CPM. It is assumed that the channel is an additive white Gaussian noise (AWGN) channel and that the receiver performs coherent maximum likelihood sequence detection.

An upper bound on the bit error probability is derived, using the generating function technique. This upper bound is valid for the digital communication system described in Part I. Thus, for any combination of rate k/n convolutional encoder, M -level mapper, frequency pulse and modulation index (rational).

For large signal-to-noise ratios this upper bound is dominated by the free Euclidean distance and by the characteristics of the signal pairs giving the free Euclidean distance.

The upper bound is evaluated numerically for a number of coded, as well as uncoded, multi-level full response CPFSK schemes.

We conclude that the free Euclidean distance is a very good performance measure for the considered class of signals.

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1. INTRODUCTION

During the last few years a large number of constant amplitude digital modulation schemes have been proposed and analyzed in the literature. Many of these schemes fit in as special cases in the continuous phase modulation (CPM) class [1]-[2]. Memory is introduced into the transmitted signal by means of continuous phase and also by means of partial response smoothing (correlative encoding). Uncoded partial response CPM yield good combined power and bandwidth efficiency [1]-[2]. Combined convolutional coding and multi-level CPM [3]-[8] is considered in this part. Thus more memory is introduced in the transmitted signal. We will only consider transmission over an additive white Gaussian noise (AWGN) channel. Furthermore, it is assumed that the receiver is an ideal coherent maximum likelihood sequence detector (MLSD, the Viterbi algorithm).

In this part, the bit error probability properties, for the schemes considered in Parts I-III, are studied. A general system description is given in Part I, and this description will be used frequently. The bit error probability, P_b , is defined as the average number of information bit errors per decoded information bit. An upper bound on P_b is derived, using the generating function technique [9]-[11]. This bound is evaluated numerically for a number of interesting coded, as well as uncoded, multi-level full response CPFSK schemes.

Similar work can be found in [12] where the symbol error probability for uncoded CPM is studied.

In Parts I-III we have used phase and state mergers to describe pair of signals. The underlying bit sequences were denoted by $\underline{u}$ and $\underline{w}$ respectively. In this part we will use the notation $\hat{\underline{u}}$ instead of $\underline{w}$, to stress that $\hat{\underline{u}}$ ($\underline{w}$) is associated with the output of the maximum likelihood receiver, see fig. 1. The notation $\underline{u}$ will still be used for the original information bit sequence. Furthermore, a phase and state merger is in the sequel referred to as an error event (an error event is defined in section III).

II. THE EUCLIDEAN DISTANCE

Let the sequences $\underline{u} = (\underline{u}'_0, \underline{u}'_1, \dots, \underline{u}'_{\ell-1})$ and $\underline{\hat{u}} = (\hat{\underline{u}}'_0, \hat{\underline{u}}'_1, \dots, \hat{\underline{u}}'_{\ell-1})$ correspond to the channel symbol sequences $\underline{\alpha}$ and $\underline{\hat{\alpha}}$ respectively. The corresponding signals $s(t, \underline{\alpha})$ and $s(t, \underline{\hat{\alpha}})$ are both assumed to be in state number s , $s \in \{1, 2, \dots, S\}$, at level 0. It is also assumed that $\underline{u}$ and $\underline{\hat{u}}$ are such that $\underline{u}'_0 \neq \hat{\underline{u}}'_0$ and such that $s(t, \underline{\alpha})$ and $s(t, \underline{\hat{\alpha}})$ are in the same state for the first time in level ℓ . In terms of the state sequences associated with $s(t, \underline{\alpha})$ and $s(t, \underline{\hat{\alpha}})$ this means that $\underline{\sigma}_0 = \hat{\underline{\sigma}}_0 = \text{state number } s$, $\underline{\sigma}_j \neq \hat{\underline{\sigma}}_j$ $\forall j$ satisfying $0 < j < \ell$, and $\underline{\sigma}_\ell = \hat{\underline{\sigma}}_\ell$. The situation described above is an example of an error event of length ℓ intervals. Error events are essential in the performance analysis of the receiver. One of the most important characteristics of an error event is its Euclidean distance. Especially, for large signal-to-noise ratios (E_b/N_0), the error probability behaviour is dominated by the minimum Euclidean distance in the set of all error events (the free Euclidean distance). The importance of error events is further clarified in section III. The normalized squared Euclidean distance, for the error event given above, is

$$d^2 = \frac{1}{2E_b} \int_0^{2N'T} \left(s(t, \underline{\alpha}) - s(t, \underline{\hat{\alpha}}) \right)^2 dt \quad (1)$$

To calculate d^2 it is convenient to use the function $d_{\text{inc}}^2(m, m_1)$, see eqs. (24)-(25) in Part I with $\underline{\gamma} = \underline{\alpha} - \underline{\hat{\alpha}}$.

To produce an error event of length ℓ intervals, $\underline{u}$ and $\underline{\hat{u}}$ must have the following structure (compare section III in Part I)

$$\begin{cases} \underline{u} = (\underline{u}'_0, \underline{u}'_1, \dots, \underline{u}'_{\ell-\eta-1}, \underline{u}'_{\ell-\eta}, \dots, \underline{u}'_{\ell-1}) \\ \underline{\hat{u}} = (\hat{\underline{u}}'_0, \hat{\underline{u}}'_1, \dots, \hat{\underline{u}}'_{\ell-\eta-1}, \underline{u}'_{\ell-\eta}, \dots, \underline{u}'_{\ell-1}); \quad \hat{\underline{u}}'_0 \neq \underline{u}'_0 \end{cases} \quad (2)$$

That is they have the same "tail", $\underline{u}'_{\ell-\eta}, \dots, \underline{u}'_{\ell-1}$ (if $\ell \geq 2$). Furthermore, the sub-sequences $\underline{u}'_0, \underline{u}'_1, \dots, \underline{u}'_{\ell-\eta-1}$ and $\hat{\underline{u}}'_0, \hat{\underline{u}}'_1, \dots, \hat{\underline{u}}'_{\ell-\eta-1}$ must both produce the same state of the convolutional encoder $x_{\ell-\eta}$. These sub-sequences must also be such that the corresponding channel symbol sub-sequences satisfy (phase merger)

$$h \cdot q(LT) \sum_{i=0}^{(\ell-\eta)N'-1} \gamma_i = n_1 \quad (3)$$

for some integer n_1 . Since the normalized squared Euclidean distance for an error event, d^2 , only depends on $\gamma_0, \gamma_1, \dots, \gamma_{(\ell-\eta)N'-1}$, it is clear that the "tail" can be replaced with zeroes without affecting d^2 (compare section III in Part I).

It is also seen that $\underline{u}$ and $\hat{\underline{u}}$ do not depend on the specific shape of the frequency-pulse $g(t)$ (for fixed L).

Suppose that $\underline{u}$ and $\hat{\underline{u}}$, eq. (2), starting in state $\underline{\sigma}_0$ generates an error event. The state of the convolutional encoder at level 0, given by $\underline{\sigma}_0$, is $\underline{x}_0$. Now take another state to start from, say $\tilde{\underline{\sigma}}_0$. Assume that $\tilde{\underline{\sigma}}_0$ is such that the state of the convolutional encoder at level 0, given by $\tilde{\underline{\sigma}}_0$, also equals $\underline{x}_0$. Then $\underline{u}$ and $\hat{\underline{u}}$, starting in state $\tilde{\underline{\sigma}}_0$, generates an error event identical (in terms of $\underline{\gamma}$) to the error event generated by $\underline{u}$, $\hat{\underline{u}}$ and $\underline{\sigma}_0$. This property will be used in section III.

III. AN UPPER BOUND ON THE BIT ERROR PROBABILITY

Assume that we are going to communicate K information bits

$\underline{u} = (u_0, u_1, \dots, u_{K-1})$. The corresponding decoded sequence produced by the receiver is $\hat{\underline{u}} = (\hat{u}_0, \hat{u}_1, \dots, \hat{u}_{K-1})$, see fig. 1. We are interested in the bit error probability P_b defined as the expected number of information bit errors per decoded information bit

$$P_b = \frac{E[w_H(\underline{u} \oplus \hat{\underline{u}})]}{K} \quad (4)$$

where $E[\cdot]$ denotes the expected value and where $w_H(\cdot)$ denotes the Hamming weight. The expectation is over all pair of sequences $(\underline{u}, \hat{\underline{u}})$. An equivalent definition of P_b is

$$P_b = \frac{1}{K} \sum_{i=0}^{K-1} \text{Prob}\{u_i \neq \hat{u}_i\} \quad (5)$$

Thus, P_b can be viewed as the average of the individual bit error probabilities $\text{Prob}\{u_i \neq \hat{u}_i\}$, $i = 0, 1, \dots, K-1$. To see this we define the random variables $x_0, x_1, \dots, x_{K-1}$ in the following way

$$x_i = \begin{cases} 1 & \text{if } u_i \neq \hat{u}_i \\ 0 & \text{if } u_i = \hat{u}_i \end{cases} \quad (6)$$

Then

$$w_H(\underline{u} \oplus \hat{\underline{u}}) = \sum_{i=0}^{K-1} x_i \quad (7)$$

and P_b , eq. (4), becomes

$$P_b = \frac{1}{K} E \left[\sum_{i=0}^{K-1} x_i \right] = \frac{1}{K} \sum_{i=0}^{K-1} E[x_i] = \frac{1}{K} \sum_{i=0}^{K-1} \text{Prob}\{u_i \neq \hat{u}_i\} \quad (8)$$

which is equal to eq. (5).

We are especially interested in P_b when the number of information bits (K) is very large, and in this section an upper bound on P_b is derived for this case. Parts of the derivation given below, is inspired from [14].

In the derivation of the upper bound on P_b it is assumed that the information bits u_i , $i = 0, 1, \dots, K-1$, are independent, identically distributed and $\text{Prob}\{u_i=1\} = \text{Prob}\{u_i=0\} = 1/2$ all i . It is also assumed that the transmission starts at level 0 in state $\underline{\sigma}_0 = (0, 0, \dots, 0)$. Furthermore, it is assumed that a sequence of K_1 "dummy-bits" is transmitted after the K information bits, to force the state of the transmitted signal to state $(0, 0, \dots, 0)$ at the end of the transmission. The total number of transmitted blocks (of length k') is therefore $B+B_1$ where $B = K/k'$ and where $B_1 = K_1/k'$.

The upper bound on P_b is then obtained by letting B (or K) be very large ($B \rightarrow \infty$).

Error Events:

The transmitters path through the trellis (the correct path) is described by the pairs $(\underline{u}'_j, \underline{\sigma}_j)$, $j = 0, 1, 2, \dots, B+B_1-1$. Similarly, the decoders path through the trellis is described by the pairs $(\hat{\underline{u}}'_j, \hat{\underline{\sigma}}_j)$, $j = 0, 1, 2, \dots, B+B_1-1$.

An error event is now defined by

i) an error event starts at level j if and only if

$$\{\hat{\underline{u}}'_j \neq \underline{u}'_j \text{ and } \hat{\underline{\sigma}}_j = \underline{\sigma}_j\}.$$

ii) an error event ends at level ℓ as soon as $\hat{\underline{\sigma}}_\ell = \underline{\sigma}_\ell$. Thus,

$$\underline{\sigma}_j = \hat{\underline{\sigma}}_j, \underline{\sigma}_i \neq \hat{\underline{\sigma}}_i \quad \forall i \text{ such that } j < i < \ell, \text{ and } \underline{\sigma}_\ell = \hat{\underline{\sigma}}_\ell.$$

From this follows that whenever a decoding error occurs, an error event must be in progress or start. An example where two error events occur is given in fig. 2. Note that some modulation schemes have error events of length 1 interval. As an example of this we have the uncoded 4-level full response CPM scheme with $h=1/2$.

Derivation of an upper bound on P_b :

The sequences $\underline{u}$ and $\hat{\underline{u}}$ (containing $K+K_1$ bits) will typically generate a sequence of error events, see fig. 3.

Let us define the random variables X_j , $j = 0, 1, \dots, B-1$, in the following way:

"If $\underline{u}$ and $\hat{\underline{u}}$ are such that an error event starts in level j , then let X_j denote the number of information bit errors given by this error event. If $\underline{u}$ and $\hat{\underline{u}}$ are such that an error event does not start in level j , then let $X_j=0$."

Then the total number of information bit errors, N_e , given by $\underline{u}$ and $\hat{\underline{u}}$ is

$$N_e = \sum_{j=0}^{B-1} X_j \quad (9)$$

and we can therefore write the bit error probability as

$$P_b = \frac{1}{K} E[N_e] = \frac{1}{K} E \left[\sum_{j=0}^{B-1} X_j \right] = \frac{1}{k \cdot B} \sum_{j=0}^{B-1} E[X_j] \quad (10)$$

To simplify the derivation we will ignore the last error event if it contains any dummy bits. The last error event in fig. 3 is an example of such an error event. Since we are interested in P_b when the number of information bits are very large, this modification is assumed to have no significant influence on P_b .

Therefore let us define the random variables W_j , $j = 0, 1, \dots, B-1$ in the following way:

"If $\underline{u}$ and $\hat{\underline{u}}$ are such that an error event starts in level j and if this error event does not contain any dummy bits, then let W_j denote the number of information bit errors given by this error event.

If $\underline{u}$ and $\hat{\underline{u}}$ are such that an error event starts in level j and if this error event contains dummy bits then let $W_j=0$.

If $\underline{u}$ and $\hat{\underline{u}}$ are such that an error event does not start in level j then let $W_j=0$."

Then the total number of information bit errors N_e , is

$$N_e \geq \sum_{j=0}^{B-1} W_j \quad (11)$$

and we can therefore write

$$P_b \geq \tilde{P}_b = \frac{1}{k \cdot B} \sum_{j=0}^{B-1} E[W_j] \quad (12)$$

It is assumed that P_b and $\tilde{P}_b$ can be made arbitrarily close to each other by letting K be large enough. We will now upper bound $\tilde{P}_b$. When K is large it is assumed that this upper bound is also an upper bound on P_b .

To find $E[W_j]$ let us study the set of all error events starting in level j for which $W_j > 0$. Take an arbitrary state, say state number s , $s \in \{1, 2, 3, \dots, S\}$. All error events which starts from this state is now numbered as $F_{s,1}, F_{s,2}, F_{s,3}, \dots$. A specific error event, say $F_{s,m}$ is completely described by its start state s and by the pair of sequences $(\underline{u}_{s,m}^C, \underline{u}_{s,m}^R)$ which generate $F_{s,m}$. The set of all error events starting in level j is then obtained by constructing $S-1$ additional such lists, one list for each state. Furthermore, define $i_{s,m}$ as the number of information bit errors associated with $F_{s,m}$, that is $i_{s,m} = w_H(\underline{u}_{s,m}^C \oplus \underline{u}_{s,m}^R)$. Define also $\ell_{s,m}$ as the length (in intervals) of $F_{s,m}$. The specific error event $F_{s,m}$ is visualized in fig. 4.

The normalized squared Euclidean distance, $d_{s,m}^2$, associated with the specific error event $F_{s,m}$ is given by (see eq. (31) in Part I)

$$d_{s,m}^2 = R \int_0^{\ell_{s,m} N'} \left(1 - \cos \left[2\pi \sum_{i=0}^{(\ell_{s,m} - \eta) N' - 1} \gamma_{jN'+i} q((x-i)T) \right] \right) dx \quad (13)$$

So far, we have studied the set of all error events starting in level j for which $W_j > 0$. Each specific member in this set, say $F_{s,m}$, ($s = 1, 2, \dots, S$; $m = 1, 2, \dots$ (m is finite)) is (partially) characterized by the three parameters $i_{s,m}$, $\ell_{s,m}$ and $d_{s,m}^2$ respectively.

Our goal is to upper bound $E[W_j]$ in eq. (12). To be able to do this we define the events $A_{j,s,m}$ and $B_{j,s,m}$, $j = 0, 1, \dots, B-1$, $s = 1, 2, \dots, S$, $m = 1, 2, \dots$ (m finite), by

$A_{j,s,m}$ = the event that $\hat{\underline{u}}_j$ is such that $\hat{\sigma}_j$ equals state number s , and such that $(\hat{\underline{u}}_j', \hat{\underline{u}}_{j+1}', \dots, \hat{\underline{u}}_{j+v}') = \underline{u}_{s,m}^R$, where $v = \ell_{s,m} - 1$.

$B_{j,s,m}$ = the event that $\underline{u}_j$ is such that σ_j equals state number s , and such that $(\underline{u}_j', \underline{u}_{j+1}', \dots, \underline{u}_{j+v}') = \underline{u}_{s,m}^C$, where $v = \ell_{s,m} - 1$.

Since the information bits are assumed to be independent and equally likely, we have

$$\text{Prob}\{B_{j,s,m}\} = \text{Prob}\{\sigma_j = \text{state no. } s\} \cdot \left(\frac{1}{2}\right)^{k' \ell_{s,m}} \quad (14)$$

The reason why we introduced the events given above, is that the joint event $\{A_{j,s,m}, B_{j,s,m}\}$ exactly describes the situation when the specific error event $F_{s,m}$ actually occurs, starting in level j . We can therefore write $E[W_j]$ as

$$\begin{aligned} E[W_j] &= \sum_i i \text{Prob}\{W_j=i\} = \sum_s \sum_m i_{s,m} \text{Prob}\{A_{j,s,m}, B_{j,s,m}\} = \\ &= \sum_s \sum_m i_{s,m} \text{Prob}\{A_{j,s,m} | B_{j,s,m}\} \cdot \text{Prob}\{\sigma_j = \text{state no. } s\} \cdot \left(\frac{1}{2}\right)^{k' \ell_{s,m}} \quad (15) \end{aligned}$$

Using the fact that $\text{Prob}\{A_{j,s,m} | B_{j,s,m}\} \leq Q\left(\sqrt{d_{s,m}^2 E_b/N_0}\right)$ (this is proved below) $E[W_j]$ is upper bounded by

$$E[W_j] \leq \sum_s \text{Prob}\{\sigma_j = \text{state no. } s\} \sum_m i_{s,m} \left(\frac{1}{2}\right)^{k' \ell_{s,m}} Q\left(\sqrt{d_{s,m}^2 E_b/N_0}\right) \quad (16)$$

where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-y^2/2} dy \quad (17)$$

Proof that $\text{Prob}\{A_{j,s,m} | B_{j,s,m}\} \leq Q\left(\sqrt{d_{s,m}^2 E_b/N_0}\right)$:

Define the event $H_{j,s,m}$ by

$H_{j,s,m}$ = the event that the received signal $r(t)$, over the unmerged span, is closer (in signal space) to the path given by $\underline{u}_{s,m}^r$ starting in level j in state s , than to the path given by $\underline{u}_{s,m}^c$ starting in level j in state s .

An equivalent definition of the event $H_{j,s,m}$ is

$H_{j,s,m}$ = the event that the metric, over the unmerged span, is larger for the path given by $\underline{u}_{s,m}^r$ starting in level j in state s , than for the path given by $\underline{u}_{s,m}^c$ starting in level j in state s .

Now note that the 'joint event $\{A_{j,s,m}, B_{j,s,m}\} \Rightarrow$ the joint event $\{H_{j,s,m}, B_{j,s,m}\}$. Thus, the joint event $\{H_{j,s,m}, B_{j,s,m}\}$ is a necessary condition for the joint event $\{A_{j,s,m}, B_{j,s,m}\}$ to occur. Suppose we know that the joint event $\{H_{j,s,m}, B_{j,s,m}\}$ has occurred, then we can not guarantee that the final choice of the Viterbi algorithm ($\hat{u}$) is such that the joint event $\{A_{j,s,m}, B_{j,s,m}\}$ occurs. Therefore, the joint event $\{H_{j,s,m}, B_{j,s,m}\}$ does not imply the joint event $\{A_{j,s,m}, B_{j,s,m}\}$. Fig. 5 illustrates the above.

Consequently, we have

$$\text{Prob}\{A_{j,s,m}, B_{j,s,m}\} \leq \text{Prob}\{H_{j,s,m}, B_{j,s,m}\} \quad (18)$$

Dividing both sides with $\text{Prob}\{B_{j,s,m}\}$ yields

$$\text{Prob}\{A_{j,s,m} | B_{j,s,m}\} \leq \text{Prob}\{H_{j,s,m} | B_{j,s,m}\} \quad (19)$$

$\text{Prob}\{H_{j,s,m} | B_{j,s,m}\}$ is a classical binary hypothesis test problem with the solution $Q\left(\sqrt{d_{s,m}^2 E_b/N_0}\right)$. Thus we have proved that

$$\text{Prob}\{A_{j,s,m} | B_{j,s,m}\} \leq Q\left(\sqrt{d_{s,m}^2 E_b/N_0}\right) \quad (20)$$

By adding terms, associated with non-existing error events, to the sum in eq. (16), the right hand side is upper-bounded by

$$\begin{aligned} E[W_j] &\leq \sum_s \text{Prob}\{\underline{\sigma}_j = \text{state no. } s\} \sum_m i_{s,m} \left(\frac{1}{2}\right)^{k'l_{s,m}} Q\left(\sqrt{d_{s,m}^2 E_b/N_0}\right) \leq \\ &\leq \sum_s \text{Prob}\{\underline{\sigma}_j = \text{state no. } s\} \cdot G_s \end{aligned} \quad (21)$$

where

$$G_s = \sum_i \sum_{\ell} \sum_d a(s, i, \ell, d) i \left(\frac{1}{2}\right)^{k'\ell} Q\left(\sqrt{d^2 E_b/N_0}\right) \quad (22)$$

and where

- $a(s, i, \ell, d)$ = the number of error events starting from state s ,
 if the trellis is **infinite**, having
 - i information bit errors
 - length $k'\ell$ information bits
 - normalized squared Euclidean distance d^2

It is seen in eq. (21) that the error events affect the upper bound through the parameters $i_{s,m}$, $\ell_{s,m}$ and $d_{s,m}^2$ respectively. The specific error event $F_{s,m}$ is generated by the sequences $\underline{u}_{s,m}^c$ and $\underline{u}_{s,m}^r$ respectively, starting in state number s in level j . Let us now study the state of the convolutional encoder, $\underline{x}_j$, which is given by state number s at level j . Now take another state at level j , say state number v . Assume that state number v is such that it yields the same state of the convolutional encoder, $\underline{x}_j$, as state number s does. Then the sequences $\underline{u}_{s,m}^c$ and $\underline{u}_{s,m}^r$, starting in state v in level j , generates error event $F_{v,m}$. Furthermore, $i_{v,m} = i_{s,m}$, $\ell_{v,m} = \ell_{s,m}$ and $d_{v,m}^2 = d_{s,m}^2$. Consequently, the two error events $F_{v,m}$ and $F_{s,m}$, $m = 1, 2, \dots$, are identical in terms of the parameters i , ℓ and d^2 , see also section II. This imply that $G_s = G_v$. Now it is clear that we can divide the S states at level j into 2^V groups. All states in a specific groupe yield the same encoder state $\underline{x}_j$ (they have the same G_s). Eq. (21) can then be written

$$E[W_j] \leq \sum_{z=1}^{2^V} \text{Prob}\{\underline{\sigma}_j \in \text{group no. } z\} \cdot \Gamma_z \quad (23)$$

where $\Gamma_z = G_w$ and where w denotes any state in group no. z . Note that Γ_z is independent of j .

Since $\text{Prob}\{\underline{\sigma}_j \in \text{group no. } z\} = 2^{-V}$ for all $j \geq \mu$ ($\mu = \max(v_1, v_2, \dots, v_k)$), and since Γ_z is independent of j , $\tilde{P}_b$ eq. (12), is upper bounded by

$$\begin{aligned} \tilde{P}_b &\leq \frac{1}{k'B} \sum_{j=0}^{B-1} \sum_{z=1}^{2^V} \text{Prob}\{\underline{\sigma}_j \in \text{group no. } z\} \cdot \Gamma_z = \\ &= \frac{1}{k'B} \sum_{j=0}^{\mu-1} \sum_{z=1}^{2^V} \text{Prob}\{\underline{\sigma}_j \in \text{group no. } z\} \cdot \Gamma_z + \frac{B-\mu}{k'B} 2^{-V} \sum_{z=1}^{2^V} \Gamma_z \quad (24) \end{aligned}$$

It is seen in eq. (24) that the first term becomes insignificant compared to the second term, if $B(K)$ is large enough. Thus, letting B be very large ($B \rightarrow \infty$) yields

$$\tilde{P}_b \leq \frac{1}{k'} \cdot 2^{-V} \sum_{z=1}^{2^V} \Gamma_z \quad (25)$$

It is assumed that this upper bound (eq. (25)) is also an upper bound on P_b . If eq. (22) is used in eq. (25) we have

$$\begin{aligned} P_b &\leq \frac{1}{k'} \cdot 2^{-V} \sum_i \sum_{\ell} \sum_d a(i, \ell, d) i \left(\frac{1}{2}\right)^{k' \ell} Q \left(\sqrt{d^2 E_b / N_0} \right) = \\ &= \sum_d C_d Q \left(\sqrt{d^2 E_b / N_0} \right) \end{aligned} \quad (26)$$

where

$$a(i, \ell, d) = \sum_{z=1}^{2^V} a(s_z, i, \ell, d)$$

where s_z denotes any state in group no. z and where

$$C_d = \frac{2^{-V}}{k'} \sum_i \sum_{\ell} a(i, \ell, d) i \left(\frac{1}{2}\right)^{k' \ell} \quad (27)$$

From eq. (26) it is seen that for large signal-to-noise ratios (E_b/N_0), the upper bound on the bit error probability is dominated by the first term in the sum

$$P_b \leq C_{d_{\min}} \cdot Q \left(\sqrt{d_{\min}^2 E_b / N_0} \right) + \text{other terms} \quad (28)$$

Eq. (26) is the principal result. To get numerical results we will use the generating function technique [9]-[11].

The generating function technique:

By using the inequality [9]

$$Q(\sqrt{x+y}) \leq Q(\sqrt{x}) e^{-y/2}, \quad x, y \geq 0 \quad (29)$$

we can upper bound Γ_z in eq. (25). Doing this with $x = d_{\min}^2 E_b/N_0$ and $y = (d^2 - d_{\min}^2) E_b/N_0$, yields

$$P_b \leq \frac{1}{k'} Q \left(\sqrt{d_{\min}^2 E_b/N_0} \right) e^{d_{\min}^2 E_b/2N_0} 2^{-V} \sum_{z=1}^{2^V} \sum_i \sum_l \sum_d a(s_z, i, l, d) i \left(\frac{1}{2} \right)^{k'l} e^{-d^2 E_b/2N_0} \quad (30)$$

Assume that we know the generating functions $T_z(D, L, I)$, $z = 1, 2, \dots, 2^V$, given by

$$T_z(D, L, I) = \sum_i \sum_l \sum_d a(s_z, i, l, d) D^{d^2} L^l I^i ; \quad z = 1, 2, \dots, 2^V \quad (31)$$

Observe that D, L, I are "dummy" variables. When generating functions are referred to, L is a "dummy" variable and **not** related to the frequency pulse $g(t)$!

Using eq. (31), eq. (30) can be written as

$$P_b \leq \frac{1}{k'} Q \left(\sqrt{d_{\min}^2 E_b/N_0} \right) e^{d_{\min}^2 E_b/2N_0} 2^{-V} \sum_{z=1}^{2^V} \frac{\partial T_z(D, L, I)}{\partial I} \bigg|_{\substack{D=e \\ L=(1/2)^{k'} \\ I=1}}^{-E_b/2N_0} \quad (32)$$

Alternatively we can write

$$P_b \leq P_u = \frac{1}{k'} Q \left(\sqrt{d_{\min}^2 E_b/N_0} \right) e^{d_{\min}^2 E_b/2N_0} \frac{\partial T(D, L, I)}{\partial I} \bigg|_{\substack{D=e \\ L=(1/2)^{k'} \\ I=1}}^{-E_b/2N_0} \quad (33)$$

where the average generating function $T(D, L, I)$ has been used

$$T(D, L, I) = 2^{-V} \sum_{z=1}^{2^V} T_z(D, L, I) \quad (34)$$

Eq. (33) is our final result and we have also introduced the notation P_u for the upper bound. When calculating P_u , the partial derivative is approximated numerically by

$$\left. \frac{\partial T(D, L, I)}{\partial I} \right|_{\substack{D=e \\ L=(1/2)^{k'} \\ I=1}} \leq \frac{I(e^{-E_b/2N_0}, (1/2)^{k'}, 1+\epsilon) - I(e^{-E_b/2N_0}, (1/2)^{k'}, 1)}{\epsilon} \quad (35)$$

where ϵ is a small ($\epsilon \ll 1$) positive number. $Q(x)$ is conveniently upper bounded by, [13],

$$Q(x) \leq \left((0.656x + 0.344\sqrt{x^2 + 5.334}) \sqrt{2\pi} \right)^{-1} e^{-x^2/2}, \quad x \geq 0 \quad (36)$$

Above, we have assumed that $T(D, L, I)$ is known. In the next section, a method to find $T(D, L, I)$ is described.

IV. THE GENERATING FUNCTION

To find $T_z(0, L, I)$, $z = 1, 2, \dots, 2^V$, we must use a method that generates the parameters d^2 , ℓ and i for all error events, given a specific start-state. The method we use is a so called pair-state description [6]. A specific pair of sequences $(\underline{u}, \hat{\underline{u}})$ is then represented as a specific path in the pair-state trellis. Furthermore, each transition in the pair-state trellis carry the information $D^{\Delta d^2} L^{\Delta i}$, where Δd^2 and Δi are the contribution to d^2 and i respectively, given by that specific transition. Observe that D, L, I are "dummy" variables. When generating functions are referred to, L is a "dummy" variable and **not** related to the frequency pulse $g(t)$!

The pair-state at level j , $\underline{\Omega}_j$, is defined by (compare eqs. (7)-(10) in Part I)

$$\underline{\Omega}_j = \begin{cases} (\Delta\theta_j, \underline{x}_{j-\eta}, \underline{u}'_{j-\eta}, \dots, \underline{u}'_{j-1}, \hat{\underline{x}}_{j-\eta}, \hat{\underline{u}}'_{j-\eta}, \dots, \hat{\underline{u}}'_{j-1}) & ; \text{ if } L \geq 2 \\ (\Delta\theta_j, \underline{x}_j, \hat{\underline{x}}_j) & ; \text{ if } L = 1 \end{cases} \quad (37)$$

where the difference-phase-state, $\Delta\theta_j$, is

$$\Delta\theta_j = \left(2\pi h q(LT) \sum_{i=-\infty}^{jN'-L} \gamma_i \right) \bmod 2\pi = \left(2\pi \frac{\ell}{p} \sum_{i=-\infty}^{jN'-L} \gamma_i \right) \bmod 2\pi \quad (38)$$

Here LT denotes the length of the frequency pulse $g(t)$.

Let $S_{\Delta\theta}$ and S_{Ω} denote the number of difference-phase-states and the number of pair-states respectively. Then $S_{\Delta\theta} \leq p$ and

$$S_{\Omega} \leq p \cdot 2^{2(v+\eta k')} \quad (39)$$

The input (at level j) to the pair-state $\underline{\Omega}_j$ is $(\underline{u}'_j, \hat{\underline{u}}'_j)$. Therefore, the number of transitions leaving $\underline{\Omega}_j$ equals $2^{2k'}$.

Start-states:

An error event can start from the pair-state $\underline{\Omega}_0$ if and only if $\underline{\Omega}_0$ has the structure

$$\underline{\Omega}_0 = (0, \underline{x}_{-\eta}, \underline{u}'_{-\eta}, \dots, \underline{u}'_{-1}, \underline{x}_{-\eta}, \underline{u}'_{-\eta}, \dots, \underline{u}'_{-1}) \quad (40)$$

It is however, for our purpose, sufficient to consider only $S_1=2^v$ such states (see section III). These S_1 pair-states are referred to as start-states and denoted by $\underline{a}_i$, $i = 1, 2, \dots, S_1$.

Merge-states:

An error event can end in the pair-state $\underline{\Omega}_j$ ($j \geq 1$) if and only if $\underline{\Omega}_j$ has the structure

$$\underline{\Omega}_j = (0, \underline{x}_{j-\eta}, \underline{u}'_{j-\eta}, \dots, \underline{u}'_{j-1}, \underline{x}_{j-\eta}, \underline{u}'_{j-\eta}, \dots, \underline{u}'_{j-1}) \quad (41)$$

These $S_3 = 2^{(v+\eta k')}$ pair-states are referred to as merge-states and denoted by $\underline{c}_i$, $i = 1, 2, \dots, S_3$.

Intermediate-states:

Those states $\underline{\Omega}_j$ ($j \geq 1$) which are not merge-states are referred to as intermediate-states and denoted by $\underline{b}_i$, $i = 1, 2, \dots, S_2$; $S_2 = S_\Omega - S_3$.

An error event of length ℓ intervals is equivalent with a path in the pair-state trellis with the following properties:

- i) the path starts from a start-state in level 0.
- ii) $\underline{u}'_0 \neq \hat{\underline{u}}'_0$
- iii) the path is in a merge-state for the first time in level ℓ .

We say that the path ends in level ℓ .

The state sequence for an error event $(\underline{\Omega}_0, \underline{\Omega}_1, \dots, \underline{\Omega}_\ell)$ is consequently such that $\underline{\Omega}_0 \in \{a_i\}$, $\underline{\Omega}_j \in \{b_i\} \forall j$ satisfying $0 < j < \ell$, and $\underline{\Omega}_\ell \in \{c_i\}$. Below we will study the set of all such paths.

Denote by $\xi_{m,z}^j$ the generating function for all paths starting in start-state $\underline{a}_z$ and which are in the intermediate-state $\underline{b}_m$ in level j . Thus

$$\xi_{m,z}^j = \sum_d \sum_i f_m(\underline{a}_z, i, j, d) D^{d^2} L^j I^i \quad (42)$$

where

$f_m(a_z, i, j, d)$ = the number of paths, starting in start-state a_z and which are in the intermediate-state b_m in level j , having (up to level j)

- the normalized squared Euclidean distance d^2
- i information bit errors

Now assume that it is possible to go from the intermediate-state b_l to the intermediate-state b_m in one step, see fig. 6.

In general there might be more than one transition from b_l to b_m (in one step). Each such transition (with corresponding Δd^2 and Δi) give the contribution $D^{\Delta d^2} L^{\Delta i} \cdot \xi_{l,z}^{j-1}$ to $\xi_{m,z}^j$. The total contribution to $\xi_{m,z}^j$, from the intermediate-state b_l , can therefore be written as $A_{m,l} \cdot \xi_{l,z}^{j-1}$. $A_{m,l}$ is the sum of all terms $D^{\Delta d^2} L^{\Delta i}$, one term for each transition from b_l to b_m (in one step). Summing over all intermediate-states yields ($j \geq 2$)

$$\xi_{m,z}^j = \sum_{l=1}^{S_2} A_{m,l} \xi_{l,z}^{j-1} \quad (43)$$

By using matrix notation, eq. (43), becomes

$$\underline{\xi}_z^j = \underline{A} \cdot \underline{\xi}_z^{j-1} ; \quad j \geq 2 \quad (44)$$

where $\underline{\xi}_z^j$ is a column vector with S_2 elements. $\underline{A}$ is an $S_2 \times S_2$ matrix with elements $A_{m,l}$. It is seen that row number m in $\underline{A}$ represents all transitions to b_m from the intermediate-states (in one step). Furthermore, column number l in $\underline{A}$ represents all transitions from b_l that goes to intermediate-states (in one step). It is also seen that the matrix $\underline{A}$ does not depend on the start-state a_z .

From eq. (44) follows that

$$\underline{\xi}_z^j = \underline{A}^{j-1} \cdot \underline{\xi}_z ; \quad j \geq 1 \quad (45)$$

where the column vector $\underline{\xi}_z$ represents all transitions from start-state a_z to the intermediate-states in one step. That is, $\xi_{m,z}$ is the sum of all terms $D^{\Delta d^2} L^{\Delta i}$, one term for each transition from start-state a_z to the intermediate-state b_m (in one step).

Administration of the error events:

Denote by $X_{m,z}^j$ ($m = 1, 2, \dots, S_3$) the generating function for all error events starting in start-state $\underline{a}_z$ and which ends in merge-state $\underline{c}_m$ in level j . In analogy with eq. (43) we have, see fig. 6,

$$X_{m,z}^j = \sum_{\ell=1}^{S_2} C_{m,\ell} \xi_{\ell,z}^{j-1} ; \quad j \geq 2 \quad (46)$$

where $C_{m,\ell}$ equals the sum of all terms $D^{\Delta d^2} L I^{\Delta i}$, one term for each transition from the intermediate-state $\underline{b}_\ell$ to the merge-state $\underline{c}_m$ (in one step). Using matrix notation, eq. (46) becomes

$$\underline{X}_z^j = \underline{C} \cdot \underline{\xi}_z^{j-1} ; \quad j \geq 2 \quad (47)$$

where $\underline{X}_z^j$ is a column vector with S_3 elements. $\underline{C}$ is an $S_3 \times S_2$ matrix with the elements $C_{m,\ell}$. It is seen that row number m in $\underline{C}$ represents all transitions to merge-state $\underline{c}_m$ from the intermediate-states (in one step). Furthermore, column number ℓ in $\underline{C}$ represents all transitions from the intermediate-state $\underline{b}_\ell$ that goes to merge-states (in one step). It is also seen that the matrix $\underline{C}$ does not depend on the start-state $\underline{a}_z$. Using eq. (45) in eq. (47) yields

$$\underline{X}_z^j = \begin{cases} \underline{C} \cdot \underline{\xi}_z^{j-1} = \underline{C} \underline{A}^{j-2} \cdot \underline{\xi}_z & ; \quad j \geq 2 \\ \underline{D}_z & ; \quad j = 1 \end{cases} \quad (48)$$

The column vector $\underline{D}_z$ represents all error events of length 1 interval, starting from start-state $\underline{a}_z$. Element number m in $\underline{D}_z$ is $D_{m,z}$ and it equals the sum of all terms $D^{\Delta d^2} L I^{\Delta i}$, one term for each transition from start-state $\underline{a}_z$ to merge-state $\underline{c}_m$ (in one step).

As we know, $\underline{X}_z^j$ represents all error events of length j intervals, starting from start-state $\underline{a}_z$. Since we are interested in all error events, starting from start-state $\underline{a}_z$, we define the column vector $\tilde{\underline{X}}_z$ by

$$\begin{aligned}
\tilde{X}_Z &= \lim_{\ell \rightarrow \infty} \sum_{j=1}^{\ell} X_Z^j = \\
&= \underline{D}_Z + \underline{C}\underline{X}_Z + \underline{C}\underline{A}\underline{X}_Z + \underline{C}\underline{A}^2\underline{X}_Z + \underline{C}\underline{A}^3\underline{X}_Z + \dots = \\
&= \underline{D}_Z + \underline{C}(\underline{I} + \underline{A} + \underline{A}^2 + \underline{A}^3 + \dots) \underline{X}_Z = \\
&= \underline{D}_Z + \underline{C}(\underline{I} - \underline{A})^{-1} \underline{X}_Z
\end{aligned} \tag{49}$$

where $\underline{I}$ is the identity matrix. In the last equality it is assumed that the series converges.

$T_Z(D, L, I)$ is the generating function for all error events starting in start-state $\underline{a}_Z$. Thus, if we multiply the column vector $\tilde{X}_Z$ from the left, by the all-one row vector $\underline{1} = (111\dots 11)$ we have found $T_Z(D, L, I)$,

$$T_Z(D, L, I) = \underline{1} \cdot \left(\underline{C}(\underline{I} - \underline{A})^{-1} \underline{X}_Z + \underline{D}_Z \right) \tag{50}$$

The average generating function $T(D, L, I)$ is then given by

$$T(D, L, I) = 2^{-V} \sum_{z=1}^{2^V} T_Z(D, L, I) = 2^{-V} \cdot \underline{1} \cdot \left(\underline{C}(\underline{I} - \underline{A})^{-1} \underline{X} + \underline{D} \right) \tag{51}$$

where

$$\underline{X} = \sum_{z=1}^{2^V} \underline{X}_Z \tag{52}$$

and

$$\underline{D} = \sum_{z=1}^{2^V} \underline{D}_Z \tag{53}$$

Eq. (51) shows a way to calculate $T(D, L, I)$. The different start-states are handled in a "parallel" fashion by $\underline{X}$ and $\underline{D}$.

VI. NUMERICAL RESULTS

The upper bound P_u , eq. (33), on the bit error probability has been calculated for some interesting coded, as well as uncoded, M-level full response CPFSK schemes ($g(t) = 1/2T$, $0 \leq t \leq T$). The average generating function $T(D,L,I)$, eq. (51), has been evaluated by truncation of the series-expansion of the matrix $(I-A)^{-1}$. The truncation-depth, for each value of E_b/N_0 , is chosen such that no significant change is observed in the numerical results when more terms in the series are added. The numerical results are given in figures showing $10\log_{10}(P_u)$ as a function of $10\log_{10}(E_b/N_0)$ [dB]. In these figures we also give the function $10\log_{10}\left(Q\left(\sqrt{d_{\min}^2 E_b/N_0}\right)\right)$. Then we can, from the figures, find (approximately) when the minimum Euclidean distance events dominates the upper bound. From these figures we can also find (approximately) the value of $C_{d_{\min}}$ in eq. (28). As a reference we often use the uncoded binary full response CPFSK scheme with $h=1/2$, MSK, which has $d_{\min}^2=2$. Therefore the function $10\log_{10}\left(Q\left(\sqrt{2E_b/N_0}\right)\right)$ is also given in the figures.

Uncoded M-level CPFSK modulation:

The upper bound on the bit error probability for uncoded binary CPFSK modulation is given in fig. 7. It is seen in this figure that the asymptotic behaviour starts roughly at levels around 10^{-2} - 10^{-3} for the considered modulation indices in the interval $0.10 \leq h \leq 0.70$. From earlier studies we know that the pair $\underline{u} = (1,0)$ and $\hat{\underline{u}} = (0,1)$ gives the minimum Euclidean distance for these modulation indices. If there are no other minimum distance pairs, then $C_{d_{\min}}$ eqs. (27)-(28) becomes

$$C_{d_{\min}} = 2 \cdot 2 \cdot \left(\frac{1}{2}\right)^2 = 1 \quad (54)$$

which is consistent with fig. 7 except when $h=1/2$ (MSK). The reason why MSK is different is that it has more minimum distance pairs. The pair $\underline{u} = (1,1)$ and $\hat{\underline{u}} = (0,0)$ is for this modulation index also a minimum distance pair. Therefore, with MSK we have

$$C_{d_{\min}} = 4 \cdot 2 \cdot \left(\frac{1}{2}\right)^2 = 2 \quad (55)$$

which is consistent with fig. 7.

The upper bound on the bit error probability for uncoded 4-level CPFSK modulation is given in figs. 8-9. Fig. 8 shows the results when the natural binary 4-level mapping rule is used and fig. 9 shows the results when the 4-level Gray code in table 1b is used.

It is seen in figs. 8-9 that the asymptotic behaviour starts roughly at levels around 10^{-4} - 10^{-5} for the considered modulation indices in the interval $0.10 \leq h \leq 0.45$. As in the binary case, those pair of sequences which yields the difference-sequence $\underline{y} = (2, -2)$ or $\underline{y} = (-2, 2)$, are minimum distance pairs for the considered modulation indices. Generally, there are $2(M-1)^2$ such pairs for uncoded M-level full response CPM schemes. Table 2 shows the 9 pairs which yields $\underline{y} = (2, -2)$ when $M=4$. Furthermore, this table shows the number of bit errors for each pair, for the mapping rules used in figs. 8-9.

If there are no other minimum distance pairs, then

$$C_{d_{\min}} = \frac{1}{2} (8 \cdot 2 + 8 \cdot 3 + 2 \cdot 4) \left(\frac{1}{2}\right)^4 = \frac{3}{2} \quad (56)$$

for the natural binary 4-level mapping rule and

$$C_{d_{\min}} = \frac{1}{2} (18 \cdot 2) \left(\frac{1}{2}\right)^4 = \frac{9}{8} \quad (57)$$

for the Gray code, see also figs. 8-9.

The upper bound on the bit error probability for uncoded 8-level CPFSK modulation is given in figs. 10-11. Fig. 10 shows the results when the natural binary 8-level mapping rule is used and fig. 11 shows the results when the 8-level Gray code in table 1c is used.

As in the 4-level case the Gray code minimizes the number of bit errors given by the $\underline{y}$ -sequence $\underline{y} = (2, -2)$. Those pair of sequences which yields the difference sequence $\underline{y} = (2, -2)$ or $\underline{y} = (-2, 2)$ are minimum distance pairs for the considered modulation indices in the interval $0.10 \leq h \leq 8/17$. If there are no more minimum distance pairs then

$$C_{d_{\min}} = \frac{1}{3} (98 \cdot 2) \left(\frac{1}{2}\right)^6 = \frac{49}{48} \quad (58)$$

for the 8-level Gray code, see fig. 11. $C_{d_{\min}}$ for the natural binary 8-level mapping rule must of course be larger than $49/48$, see fig. 10.

It is also seen in figs. 10-11 that the difference between the upper bound and the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$ is larger for the modulation indices $h=8/19$ and $h=8/17$. For these modulation indices, the asymptotic behaviour of the upper bounds starts at much lower values on the bit error probability. The reason for this is that $h=8/19$ is very close to $h=2/5$ which is a weak modulation index with $d_{\min}^2(2/5)=3$, and that $h=8/17$ is very close to $h=1/2$ which also is a weak modulation index with $d_{\min}^2(1/2)=3$. Furthermore, for both $h=2/5$ and for $h=1/2$ all minimum distance pairs have length $1T$ (error events with length 1 interval). When the modulation index h is very close to a weak modulation index, as described above, there exist many pair of signals with phase functions very close to each other. This implies that there exist many error events with distances which differ only with a small amount. Thus, the distance structure is different when the modulation index h is very close to a weak modulation index. The larger difference between the upper bound and the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$ is the result of this change in distance structure. Thus, the effect of the "other terms" in eq. (28) is still significant for the values of E_b/N_0 studied in figs. 10-11.

Fig.12 shows the upper bound on the bit error probability for some interesting uncoded M-level CPFSK schemes. It is seen that with $M=16$ and $h=8/17$, we have nearly a 6dB asymptotic gain compared to MSK.

In general it is not so easy to find $C_{d_{\min}}$. If however, those pair of sequences which yields the difference sequence $\underline{y} = (2, -2)$ or $\underline{y} = (-2, 2)$ are the only existing minimum distance pairs, then

$$C_{d_{\min}} \geq \frac{1}{k'} \left(2 \cdot 2^{(M-1)} \right)^2 \left(\frac{1}{2} \right)^{2k'} = \frac{4}{\log_2(M)} \cdot \left(\frac{M-1}{M} \right)^2 \quad (59)$$

for uncoded M-level full response schemes. Examples of both 4-level and 8-level CPFSK schemes, which achieve this lower bound on $C_{d_{\min}}$, have been given above (the Gray mappers). Note that $C_{d_{\min}}$ can be made less than 1. If for example uncoded 16-level (Gray map.) CPFSK modulation is used, then $C_{d_{\min}} = (15/16)^2 < 1$. This is illustrated in fig. 13 for the modulation indices $h=0.10$ and $h=0.20$ respectively.

Rate 1/2 convolutional encoder and binary CPFSK modulation:

The upper bound on the bit error probability has been evaluated for some of the coded binary CPFSK schemes found in Part II. Fig. 14 shows the result for the (1,2)-encoder. For this scheme with $h=0.20$, all minimum distance pairs have length $6T$. Furthermore, there are 8 such pairs, and each pair gives 2 bit errors. Consequently, $C_{d_{\min}}$ can be calculated for the scheme in fig. 14, with the modulation index $h=0.20$,

$$C_{d_{\min}} = 2^{-1} \left(8 \cdot 2 \cdot \left(\frac{1}{2} \right)^3 \right) = 1 \quad (60)$$

which is consistent with fig. 14. Note that $C_{d_{\min}}$ is larger when $h=1/4$ and this is due to additional minimum distance pairs caused by modulo- 2π effects. Fig. 15 show the results for the (2,1)-encoder and figs. 16-17 considers the (4,3)-encoder. For the scheme in figs. 16-17, with $h=0.20$, there are 32 minimum distance pairs of length $14T$ and 64 minimum distance pairs of length $16T$. Furthermore, each pair gives 4 bit errors. Thus, for the scheme in figs. 16-17, with $h=0.20$, $C_{d_{\min}}$ can be calculated to be,

$$C_{d_{\min}} = 2^{-2} \left(32 \cdot 4 \cdot \left(\frac{1}{2} \right)^7 + 64 \cdot 4 \cdot \left(\frac{1}{2} \right)^8 \right) = \frac{1}{2} \quad (61)$$

When $h=1/2$ (for the scheme in figs. 16-17) $C_{d_{\min}}$ is much larger. For this modulation index minimum distance pairs do also exist with lengths $8T$, $10T$ and $12T$. There are also additional pairs (compared to $h=0.20$) due to modulo- 2π effects.

From Part II we know that $h=1/2$ is especially interesting, since large free Euclidean distance values have been found for this modulation index. Figs. 18-19 show the upper bound for 3 good rate 1/2 encoded binary CPFSK schemes with $h=1/2$. For the (2,3)-encoder in these figures, (with $h=1/2$) there are 16 minimum distance pairs of length $6T$ and 32 minimum distance pairs of length $8T$. Furthermore, each pair gives 2 bit errors. Thus, $C_{d_{\min}}$ can be calculated for the (2,3)-encoder in figs. 18-19,

$$C_{d_{\min}} = 2^{-1} \left(16 \cdot 2 \cdot \left(\frac{1}{2} \right)^3 + 32 \cdot 2 \cdot \left(\frac{1}{2} \right)^4 \right) = 4 \quad (62)$$

For the (31,17)-encoder in figs. 18-19, the minimum distance pairs have lengths $12T$, $14T$, $16T$, $20T$ and $24T$. $C_{d_{\min}}$ for this scheme has not been calculated however.

Rate 1/2 convolutional encoder and 4-level CPFSK modulation:

The upper bound on the bit error probability has been evaluated for some of the coded 4-level CPFSK schemes found in Part II. Fig. 20 shows the result for the (1,2)-encoder. For this scheme, with $h=1/4$, all minimum distance pairs have length $3T$. Furthermore, there are 8 such pairs and each pair gives 2 bit errors. Thus, $C_{d_{\min}}$ can be calculated for the scheme in fig. 20, with $h=1/4$,

$$C_{d_{\min}} = 2^{-1} \left(8 \cdot 2 \cdot \left(\frac{1}{2} \right)^3 \right) = 1 \quad (63)$$

which is consistent with fig. 20. The (7,2)-encoder is considered in fig. 21. It is seen in this figure that $C_{d_{\min}}$ is much less than 1. For this scheme, with $h=1/10$ there are 4 minimum distance pairs, all with length $6T$, and each pair gives 2 bit errors. Note that no minimum distance pair have been found when the start-state in the convolutional encoder is $\boxed{00}$ or $\boxed{11}$. Thus, the number of minimum distance pairs are not the same for the 2^V start-states. $C_{d_{\min}}$ for the scheme in fig. 21, with $h=1/10$, can now be found,

$$C_{d_{\min}} = 2^{-2} \left(4 \cdot 2 \cdot \left(\frac{1}{2} \right)^6 \right) = 2^{-5} \quad (64)$$

When $h=1/4$ (in fig. 21) all minimum distance pairs also have length $6T$, but there exist pairs which are not minimum distance pairs for $h=1/10$. See fig. 63 in Part II.

From Part II we know that $h=1/4$ and $h=1/2$ are especially interesting, since large free Euclidean distances have been found for these modulation indices. Figs. 22-23 show the upper bound P_U for 4 good rate 1/2 encoded 4-level CPFSK schemes with $h=1/4$. For the (13,4)-encoder in these figures ($h=1/4$) all minimum distance pairs have length $8T$. Furthermore, there are 64 such pairs, and each pair gives 4 bit errors. It is also found that minimum distance pairs do not exist for the start-states $\boxed{0000}$, $\boxed{0001}$, $\boxed{1110}$ and $\boxed{1111}$ in the encoder. For the (13,4)-encoder in figs. 22-23, with $h=1/4$, $C_{d_{\min}}$ can be calculated to be,

$$C_{d_{\min}} = 2^{-3} \left(64 \cdot 4 \cdot \left(\frac{1}{2} \right)^8 \right) = 2^{-3} \quad (65)$$

For the (23,10)-encoder in figs. 22-23 (with $h=1/4$) all minimum distance pairs have length $6T$. There are 256 such pairs and each pair gives 2 bit errors. Furthermore, for this scheme, minimum distance pairs do not exist for the start-states $\boxed{0000}$, $\boxed{0001}$, $\boxed{0110}$, $\boxed{0111}$, $\boxed{1000}$, $\boxed{1001}$, $\boxed{1110}$ and $\boxed{1111}$ in the encoder. $C_{d_{\min}}$ for this scheme can now be calculated,

$$C_{d_{\min}} = 2^{-4} \left(256 \cdot 2 \cdot \left(\frac{1}{2} \right)^6 \right) = \frac{1}{2} \quad (66)$$

It is seen in figs. 22-23 that significant gains in E_b/N_0 can be obtained, compared with the uncoded MSK scheme or compared with the uncoded 4-level CPFSK scheme with $h=1/4$.

Figs. 24-25 show the upper bound P_u for 3 good rate $1/2$ encoded 4-level CPFSK schemes with $h=1/2$. However, the modulation index $h=1/4$ is to be preferred over $h=1/2$ since $h=1/4$ is more bandwidth-efficient.

High rate convolutional encoder and M-level CPFSK modulation:

The upper bound on the bit error probability has been evaluated for some of the coded 8-level and 16-level CPFSK schemes found in Part III. Fig. 26 shows the results for the rate $2/3$ (2,1)-encoder when combined with 8-level (nat.bin.) CPFSK modulation. For this scheme, with $h=1/10$, all minimum distance pairs have length $3T$. Furthermore, there are 16 such pairs, and each pair gives 4 bit errors. Thus, $C_{d_{\min}}$ can be found,

$$C_{d_{\min}} = \frac{2^{-1}}{2} \left(16 \cdot 4 \cdot \left(\frac{1}{2} \right)^6 \right) = \frac{1}{4} \quad (67)$$

The rate $2/3$ (5,2)-encoder is considered in fig. 27. For this scheme, with $h=1/10$, all minimum distance pairs have length $4T$. For $h=1/5$, in fig. 27, there is another set of minimum distance pairs. For this h ($1/5$) all minimum distance pairs have length $3T$. Furthermore, for $h=1/5$ in fig. 27, the asymptotic behaviour of the upper bound is reached for very low values on the bit error probability. The reason for this is that $d_{\min}^2(1/5) = 4.4636$ and that there exist pairs with distance equal to 4.7568 (the encoder-independent upper bound in Part III). Thus, large values on E_b/N_0 is needed, to make the effect of the "other terms" in eq. (28) insignificant.

The rate $2/3$ (15,2)-encoder is considered in figs. 28-29. For the modulation indices in this figure, all minimum distance pairs have length $5T$. Furthermore, for $h=1/7$ and for $h=2/13$ there are pairs with distances very close to $d_{\min}^2$. When $h=1/7$ there exist pairs of length $4T$ with distance 4.1576 which is close to $d_{\min}^2(1/7) = 4.1320$, and when $h=2/13$ there exist pairs of length $2T$ with distance 4.6860 (the encoder-independent upper bound in Part III)' which is close to $d_{\min}^2(2/13) = 4.6733$. By varying the modulation index in the interval $1/10 \leq h \leq 2/13$ various combinations of power saving and bandwidth saving schemes are obtained, see also the $v=3$ results in fig. 19 in Part III.

Figs. 30-31 show the results for the rate $3/4$ (5,2)-encoder when combined with 16-level (nat.bin.) CPFSK modulation. Also this scheme has good combined power and bandwidth saving properties when the modulation index is in the interval $1/11 \leq h \leq 2/13$. For this scheme, with $h=1/10$, all minimum distance pairs have length $4T$. When $h=2/13$ this scheme reaches the encoder-independent upper bound in Part III. For this h (2/13) there are 128 minimum distance pairs, where each pair has length $2T$ and has 2 bit errors. Thus, for the scheme in figs. 30-31, with the modulation index $h=2/13$,

$C_{d_{\min}}$ is

$$C_{d_{\min}} = \frac{2^{-2}}{3} \left(128 \cdot 2 \cdot \left(\frac{1}{2} \right)^6 \right) = \frac{1}{3} \quad (68)$$

If the encoder-independent upper bound on $d_{\min}^2$, given in Part III, is reached, then $C_{d_{\min}}$ can be calculated quite easily. This is done in appendix A. It is found that, with rate $2/3$ encoders and 8-level CPFSK, there are 8 minimum distance pairs for each start-state in the encoder. Furthermore, each such pair yields 2 bit errors and have length $2T$. Thus, if the encoder-independent upper bound is reached then

$$C_{d_{\min}} = \frac{2^{-v}}{2} \left(2^v \cdot 8 \cdot 2 \cdot \left(\frac{1}{2} \right)^4 \right) = \frac{1}{2} \quad (69)$$

It is also found in appendix A that, with rate $3/4$ encoders and 16-level CPFSK, $C_{d_{\min}}$ can have different values for different modulation indices. This is so since the minimum distance pairs corresponds to $\underline{y} = (8, -8)$, $\underline{y} = (16, -16)$ or $\underline{y} = (24, -24)$. Thus, from appendix A, if the encoder-independent upper bound is reached, then

$$C_{d_{\min}} = \begin{cases} \frac{2^{-v}}{3} \left(2^v \cdot 32 \cdot 2 + 2^v \cdot 32 \cdot 3 + 2^v \cdot 8 \cdot 4 \right) \left(\frac{1}{2} \right)^6 = 1, & \text{if } \underline{\gamma} = (8, -8) \\ \frac{2^{-v}}{3} \left(2^v \cdot 32 \cdot 2 \cdot \left(\frac{1}{2} \right)^6 \right) = \frac{1}{3}, & \text{if } \underline{\gamma} = (16, -16) \quad (70) \\ \frac{2^{-v}}{3} \left(2^v \cdot 8 \cdot 4 \cdot \left(\frac{1}{2} \right)^6 \right) = \frac{1}{6}, & \text{if } \underline{\gamma} = (24, -24) \end{cases}$$

Eqs. (69)-(70) is illustrated in fig. 32. All schemes considered in this figure have reached the encoder-independent upper bound on $d_{\min}^2$, given in Part III. Both the rate $2/3$ (5,2)-encoder (with $h=2/9$) and the rate $2/3$ (15,2)-encoder (with $h=1/5$) yield $C_{d_{\min}}=1/2$, according to eq. (69). The rate $3/4$ (5,2)-encoder in fig. 32, with $h=2/13$, have $C_{d_{\min}}=1/3$ since the minimum distance pairs correspond to $\underline{\gamma} = (16, -16)$. If this scheme is used with the modulation index $h=1/5$ instead, then $C_{d_{\min}}$ is smaller as can be seen in fig. 32. The reason is that for this modulation index ($h=1/5$), the rate $3/4$ (5,2)-encoder combined with 16-level CPFSK modulation, have minimum distance pairs corresponding to $\underline{\gamma} = (24, -24)$. Thus, $C_{d_{\min}}=1/6$, see eq. (70).

Simulation results:

Computer simulations have been done for a few selected schemes. The bit error probability is estimated by dividing the number of bit errors simulated, with the total number of information bits.

The variance of the estimated bit error probability depends on the number of bit errors simulated. For high bit error probabilities, 1000 bit errors have been used. For lower bit error probabilities, fewer bit errors are used because of the time consuming simulations. The memory for the detector has in all the simulations been chosen to 50 channel symbol intervals, which is large enough to ensure that all unmerged pairs have a distance greater than $d_{\min}^2$.

Simulations have been done for a few selected schemes, and these are given in figs. 33-40. The upper bound P_u and the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$ are also given in these figures. The number of bit errors simulated are also given, if it does not equal 1000. Figs. 33-35 consider uncoded binary, 4-

level and 8-level CPFSK modulation respectively. It is seen in these figures that the simulated results are good complements to the upper bound for high values on P_b .

The rate $1/2$ (7,2)-encoder combined with 4-level (nat.bin.) CPFSK modulation ($h=1/10$) is considered in fig. 36. The simulated bit error probabilities, in this figure, indicate that the upper bound is good for signal to noise ratios larger than ≈ 10 -11dB. Figs. 37-38 show the simulation results for the rate $1/2$ (13,4)-encoder and for the rate $1/2$ (23,10)-encoder respectively, when combined with 4-level CPFSK modulation and $h=1/4$.

The rate $2/3$ (2,1)-encoder combined with 8-level CPFSK modulation and $h=1/5$ is considered in fig. 39, and the rate $3/4$ (5,2)-encoder combined with 16-level CPFSK modulation is considered in fig. 40.

The simulation results given above seems to be rather accurate as long as the bit error probability is larger than $\approx 10^{-2}$ - 10^{-3} . For smaller values on P_b the simulations are very time-consuming, and our results are less reliable. This can also be seen in the figures, where the number of simulated bit errors are given.

From figs. 7-40 it is clear that the upper bound P_u , normally, is bad for large values on P_b . Large bit error probabilities are however rather easy to simulate. The combination of simulated bit error probabilities (for large P_b), and the upper bound P_u , gives a good description of the true bit error probability.

VII. DISCUSSION AND CONCLUSIONS

Modulation schemes consisting of a rate k/n convolutional encoder combined with M-level CPM is considered in this part. It is assumed that the channel is an additive white Gaussian noise channel and it is also assumed that the receiver is an ideal coherent maximum likelihood receiver (the Viterbi algorithm).

An upper bound on the bit error probability, using the generating function technique, is derived. This upper bound is valid for the digital communication system described in Part I. Thus, for any combination of rate k/n convolutional encoder, M-level mapper, frequency pulse and modulation index (rational).

Asymptotically (large E_b/N_0), this upper bound approaches the function $C_{d_{\min}} \cdot Q(\sqrt{d_{\min}^2 E_b/N_0})$, where $C_{d_{\min}}$ is a constant independent of E_b/N_0 , see section III.

The average generating function $I(D, L, I)$ is found by constructing a so called pair-state trellis [6]. The size of the matrices involved, when calculating $I(D, L, I)$, is determined by the number of intermediate-states in the pair-state trellis, see section IV.

The upper bound on the bit error probability has been evaluated numerically for some interesting coded, as well as uncoded, M-level full response CPFSK schemes. Simulation results for a few selected schemes are also given.

We conclude that the free Euclidean distance, $d_{\min}^2$, is a very good performance measure for the considered class of signals.

APPENDIX A: $C_{d_{\min}}$ FOR A CLASS OF CODED MULTI-LEVEL CPFSK SCHEMES

In this appendix we study the sub-class of schemes considered in Part III, with $n=3$ and $n=4$. Especially, $C_{d_{\min}}$ is found if the encoder-independent upper bound (eq. (5) in Part III) is reached.

Rate 2/3 convolutional encoder combined with 8-level CPFSK modulation:



If the encoder-independent upper bound is reached, then we know that all minimum distance pairs yield the difference sequence $\underline{\gamma} = (8, -8)$ or $\underline{\gamma} = (-8, 8)$. To produce the difference symbol $\gamma_0=8$, $\underline{u}_0$ and $\underline{w}_0$ must satisfy,

$$\begin{cases} \underline{u}_0 = (1, z) \\ \underline{w}_0 = (0, z) \end{cases} \quad (A1)$$

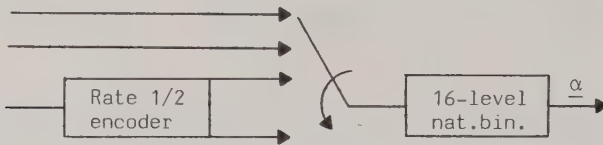
where $z \in \{0, 1\}$. Eq. (A1) follows from the properties of the natural binary 8-level mapping rule, see table 1c. Note that $\underline{u}_0$ and $\underline{w}_0$ above will produce the difference symbol $\gamma_0=8$ regardless of the start-state in the encoder. Therefore, all minimum distance pairs, associated with the difference sequence $\underline{\gamma} = (8, -8)$, can be written as

$$\begin{cases} \underline{u} = (1, z_1, 0, z_2) \\ \underline{w} = (0, z_1, 1, z_2) \end{cases} \quad (A2)$$

Thus, there are 4 such pairs, and each pair has two bit errors and length $2T$. $C_{d_{\min}}$ can now be found,

$$C_{d_{\min}} = \frac{2^{-V}}{2} \left(2^V \cdot 8 \cdot 2 \cdot \left(\frac{1}{2} \right)^4 \right) = \frac{1}{2} \quad (A3)$$

Rate 3/4 convolutional encoder combined with 16-level CPFSK modulation:



If the encoder-independent upper bound is reached, then we know that all minimum distance pairs yield the difference sequence $\underline{\gamma} = (8, -8)$, $\underline{\gamma} = (-8, 8)$, $\underline{\gamma} = (16, -16)$, $\underline{\gamma} = (-16, 16)$, $\underline{\gamma} = (24, -24)$ or $\underline{\gamma} = (-24, 24)$.

To produce the difference symbol $\gamma_0=8$, there are three possibilities

$$\left\{ \begin{array}{l} \underline{u}_0 = (1, 1, z) \\ \underline{w}_0 = (1, 0, z) \end{array} \right\}; \left\{ \begin{array}{l} \underline{u}_0 = (1, 0, z) \\ \underline{w}_0 = (0, 1, z) \end{array} \right\}; \left\{ \begin{array}{l} \underline{u}_0 = (0, 1, z) \\ \underline{w}_0 = (0, 0, z) \end{array} \right\} \quad (A4)$$

Therefore, the pairs given in the table below will produce the difference sequence $\underline{\gamma} = (8, -8)$,

$\frac{\underline{u}}{\underline{w}}$	Number of inf. bit errors
$1, 1, z_1, 1, 0, z_2$ $1, 0, z_1, 1, 1, z_2$	2
$1, 1, z_1, 0, 1, z_2$ $1, 0, z_1, 1, 0, z_2$	3
$1, 1, z_1, 0, 0, z_2$ $1, 0, z_1, 0, 1, z_2$	2
$1, 0, z_1, 1, 0, z_2$ $0, 1, z_1, 1, 1, z_2$	3
$1, 0, z_1, 0, 1, z_2$ $0, 1, z_1, 1, 0, z_2$	4
$1, 0, z_1, 0, 0, z_2$ $0, 1, z_1, 0, 1, z_2$	3
$0, 1, z_1, 1, 0, z_2$ $0, 0, z_1, 1, 1, z_2$	2
$0, 1, z_1, 0, 1, z_2$ $0, 0, z_1, 1, 0, z_2$	3
$0, 1, z_1, 0, 0, z_2$ $0, 0, z_1, 0, 1, z_2$	2

Thus, if all minimum distance pairs yield $\underline{\gamma} = (8, -8)$ or $\underline{\gamma} = (-8, 8)$, $C_{d_{\min}}$ can be found,

$$C_{d_{\min}} = \frac{2^{-V}}{3} \left(2^V \cdot 32 \cdot 2 + 2^V \cdot 32 \cdot 3 + 2^V \cdot 8 \cdot 4 \right) \left(\frac{1}{2} \right)^6 = 1 \quad (A5)$$

To produce the difference symbol $\gamma_0=16$, there are two possibilities,

$$\begin{cases} \underline{u}_0 = (1, 1, z) \\ \underline{w}_0 = (0, 1, z) \end{cases} ; \quad \begin{cases} \underline{u}_0 = (1, 0, z) \\ \underline{w}_0 = (0, 0, z) \end{cases} \quad (A6)$$

Therefore, the pairs given in the table below will produce the difference sequence $\underline{\gamma} = (16, -16)$,

$\frac{\underline{u}}{\underline{w}}$	Number of inf. bit errors
$1, 1, z_1, 0, 1, z_2$ $0, 1, z_1, 1, 1, z_2$	2
$1, 1, z_1, 0, 0, z_2$ $0, 1, z_1, 1, 0, z_2$	2
$1, 0, z_1, 0, 1, z_2$ $0, 0, z_1, 1, 1, z_2$	2
$1, 0, z_1, 0, 0, z_2$ $0, 0, z_1, 1, 0, z_2$	2

Thus, if all minimum distance pairs yield $\underline{\gamma} = (16, -16)$ or $\underline{\gamma} = (-16, 16)$, $C_{d_{\min}}$ can be found

$$C_{d_{\min}} = \frac{2^{-V}}{3} \left(2^V \cdot 32 \cdot 2 \cdot \left(\frac{1}{2} \right)^6 \right) = \frac{1}{3} \quad (A7)$$

To produce the difference symbol $\gamma_0=24$, $\underline{u}_0$ and $\underline{w}_0$ must satisfy

$$\begin{cases} \underline{u}_0 = (1, 1, z) \\ \underline{w}_0 = (0, 0, z) \end{cases} \quad (A8)$$

Thus, there exists only 4 minimum distance pairs associated with $\underline{Y} = (24, -24)$. For this case, $C_{d_{\min}}$ is

$$C_{d_{\min}} = \frac{2^{-v}}{3} \left(2^v \cdot 8 \cdot 4 \cdot \left(\frac{1}{2} \right)^6 \right) = \frac{1}{6} \quad (A9)$$

Input to the mapper	Output channel symbol
1	1
0	-1

a) M=2

Input to the mapper		Output channel symbol	
MSB	LSB	Natural binary	Gray mapper
1	1	3	1
1	0	1	3
0	1	-1	-1
0	0	-3	-3

b) M=4

Input to the mapper			Output channel symbol	
MSB		LSB	Natural binary	Gray mapper
1	1	1	7	5
1	1	0	5	7
1	0	1	3	3
1	0	0	1	1
0	1	1	-1	-5
0	1	0	-3	-7
0	0	1	-5	-3
0	0	0	-7	-1

c) M=8

Input to the mapper				Output channel symbol	
MSB			LSB	Natural binary	Gray mapper
1	1	1	1	15	5
1	1	1	0	13	7
1	1	0	1	11	3
1	1	0	0	9	1
1	0	1	1	7	11
1	0	1	0	5	9
1	0	0	1	3	13
1	0	0	0	1	15
0	1	1	1	-1	-5
0	1	1	0	-3	-7
0	1	0	1	-5	-3
0	1	0	0	-7	-1
0	0	1	1	-9	-11
0	0	1	0	-11	-9
0	0	0	1	-13	-13
0	0	0	0	-15	-15

d) M=16

Table 1. M-level mapping rules.

- a) M=2
- b) M=4
- c) M=8
- d) M=16

$\frac{\alpha}{\wedge}$	The number of inf. bit errors	
	mapping rule	
	nat. bin.	Gray map.
(3,1) (1,3)	2	2
(3,-1) (1,1)	3	2
(3,-3) (1,-1)	2	2
(1,1) (-1,3)	3	2
(1,-1) (-1,1)	4	2
(1,-3) (-1,-1)	3	2
(-1,1) (-3,3)	2	2
(-1,-1) (-3,1)	3	2
(-1,-3) (-3,-1)	2	2

Table 2. Minimum distance pairs for uncoded 4-level full response CPM, see section V.

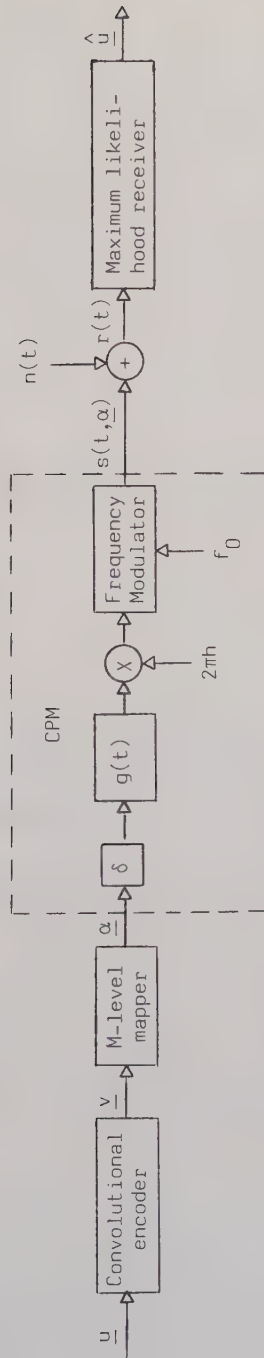


Figure 1. The modulation scheme, channel and receiver.

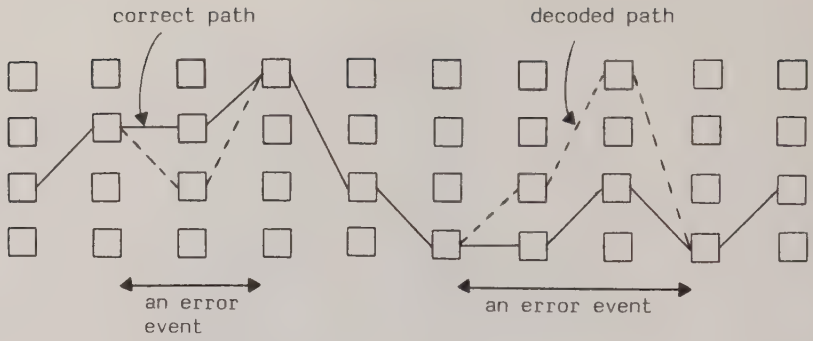


Figure 2. An example illustrating two error events.

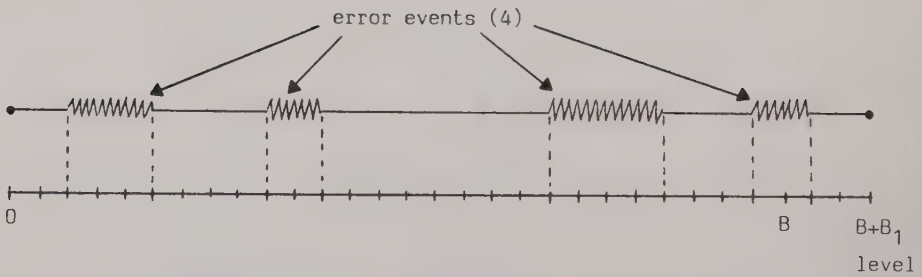


Figure 3. An example illustrating a sequence of error events.

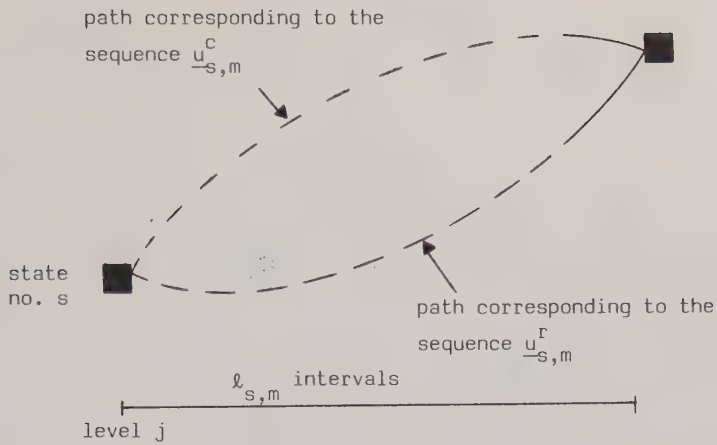


Figure 4. Illustrating the specific error event $F_{s,m}$ starting in level j, see section III for details.

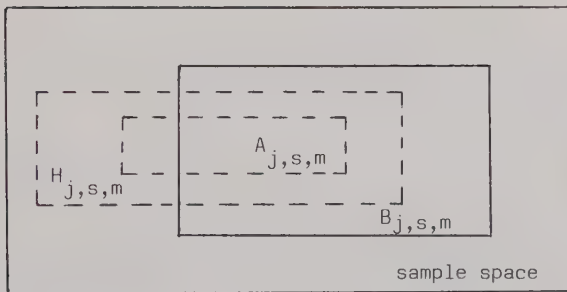


Figure 5. Illustrating 3 events used in the proof in section III.

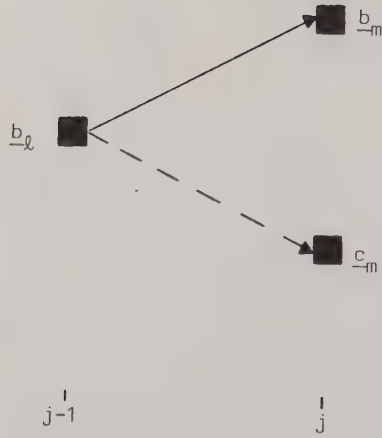


Figure 6. Illustrates transitions between intermediate-states, and between intermediate-states and merge-states, see section IV for details.

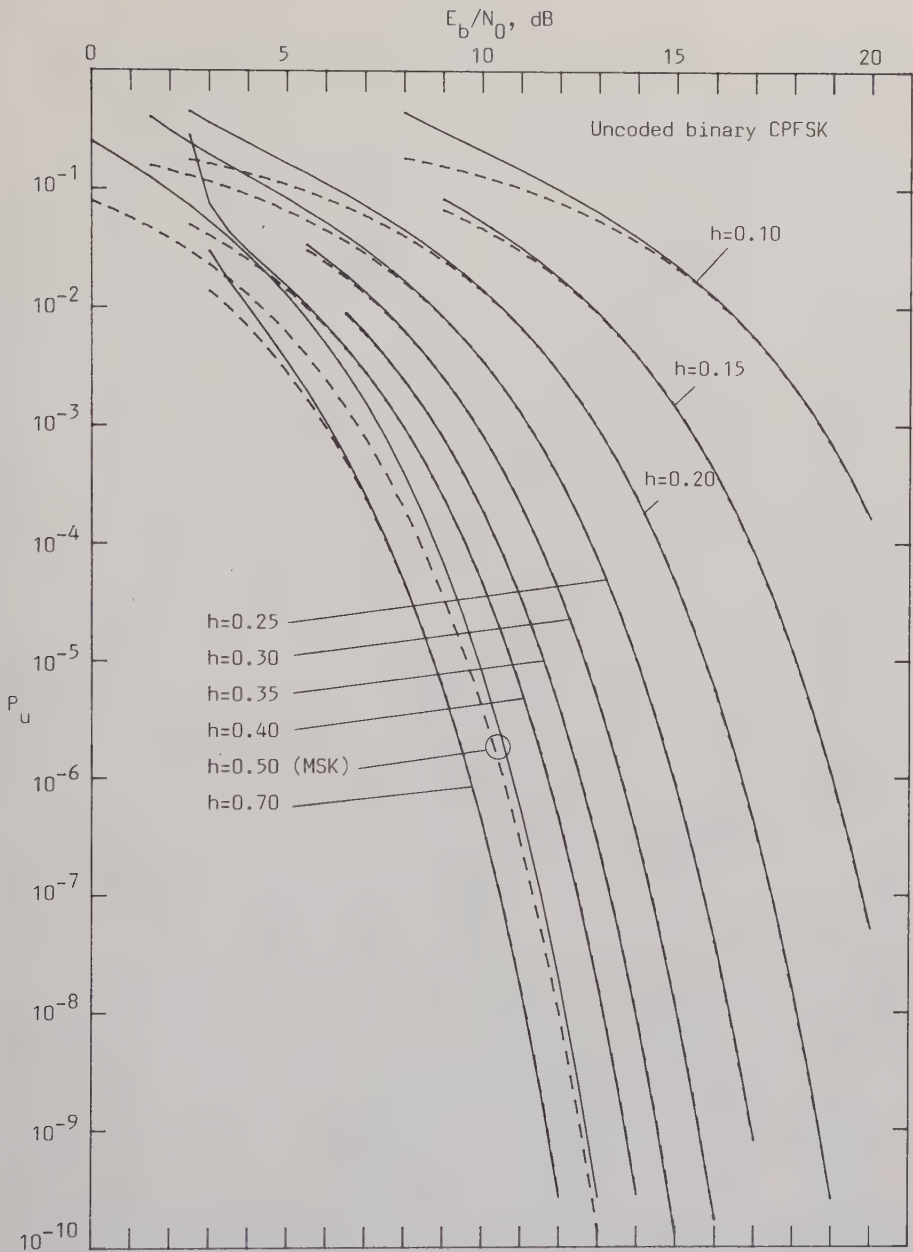


Figure 7. The upper bound P_u (solid line) on the bit error probability for uncoded binary CPFSK modulation. The dashed line is the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$.

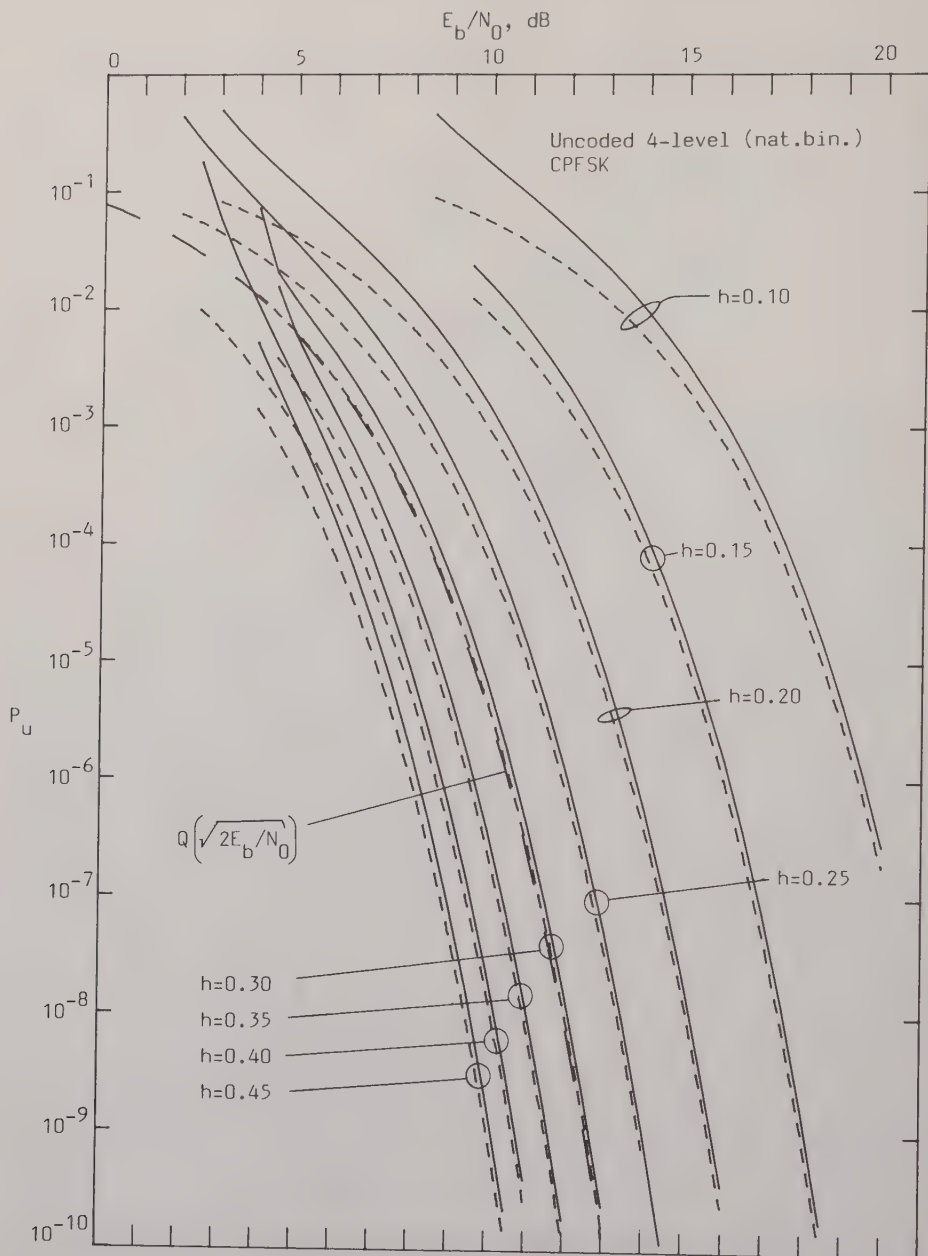


Figure 8. The upper bound P_u (solid line) on the bit error probability for uncoded 4-level (natural binary) CPFSK modulation. The dashed line is the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$.

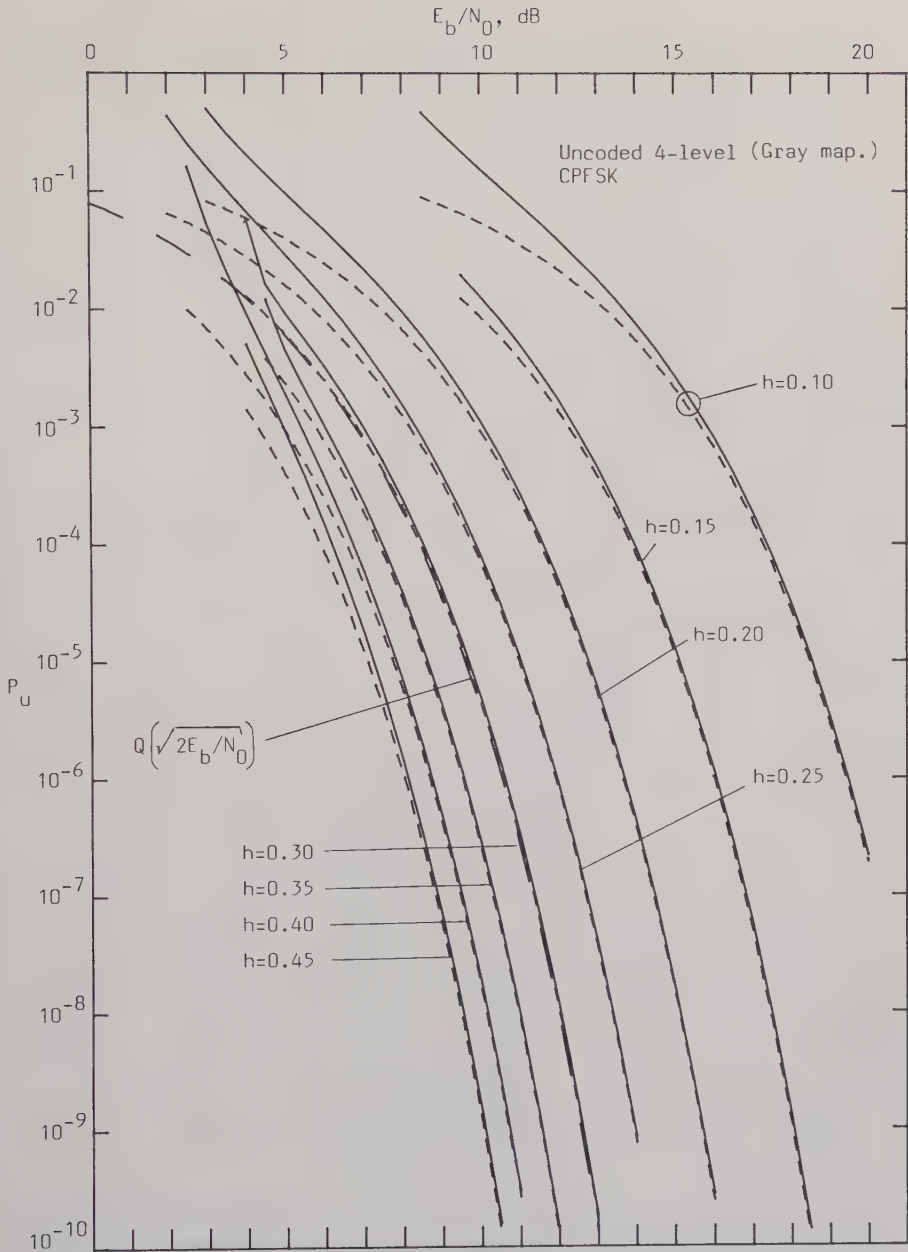


Figure 9. The upper bound P_u (solid line) on the bit error probability for uncoded 4-level (Gray map.) CPFSK modulation. The dashed line is the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$.

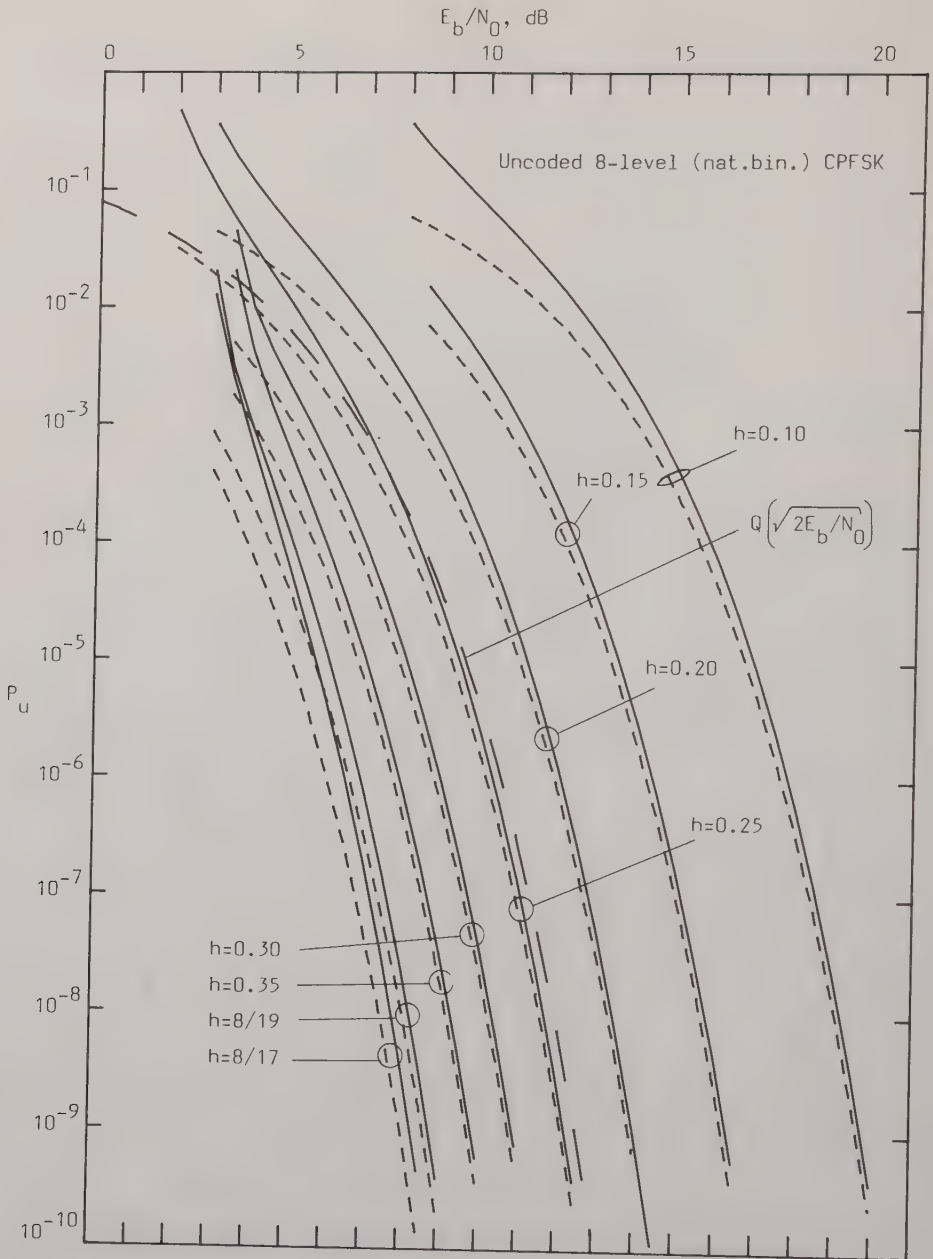


Figure 10. The upper bound P_u (solid line) on the bit error probability for uncoded 8-level (natural binary) CPFSK modulation. The dashed line is the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$.

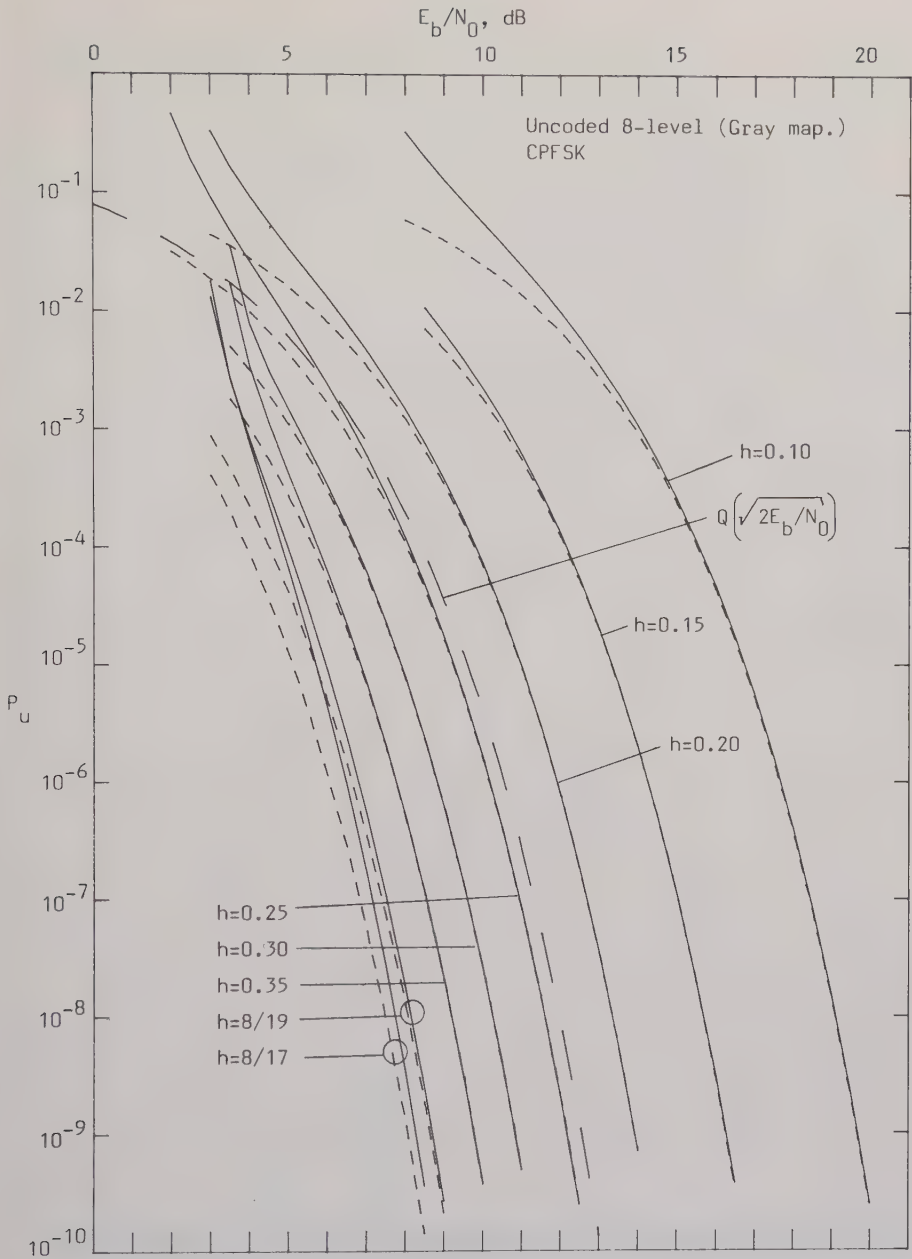


Figure 11. The upper bound P_u (solid line) on the bit error probability for uncoded 8-level (Gray map.) CPFSK modulation. The dashed line is the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$.

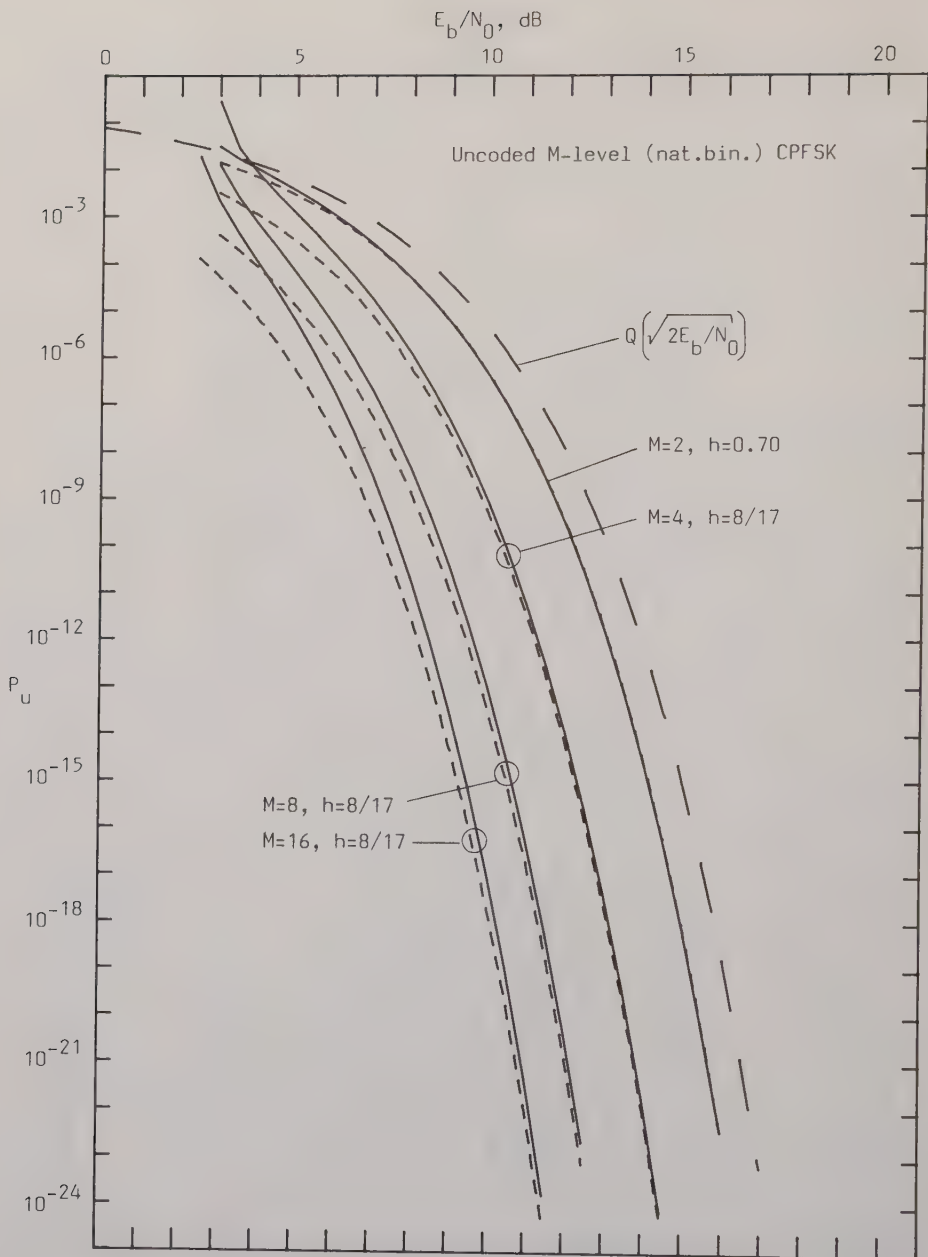


Figure 12. The upper bound P_u (solid line) on the bit error probability for some uncoded M -level (natural binary) CPFSK schemes. The dashed line is the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$.

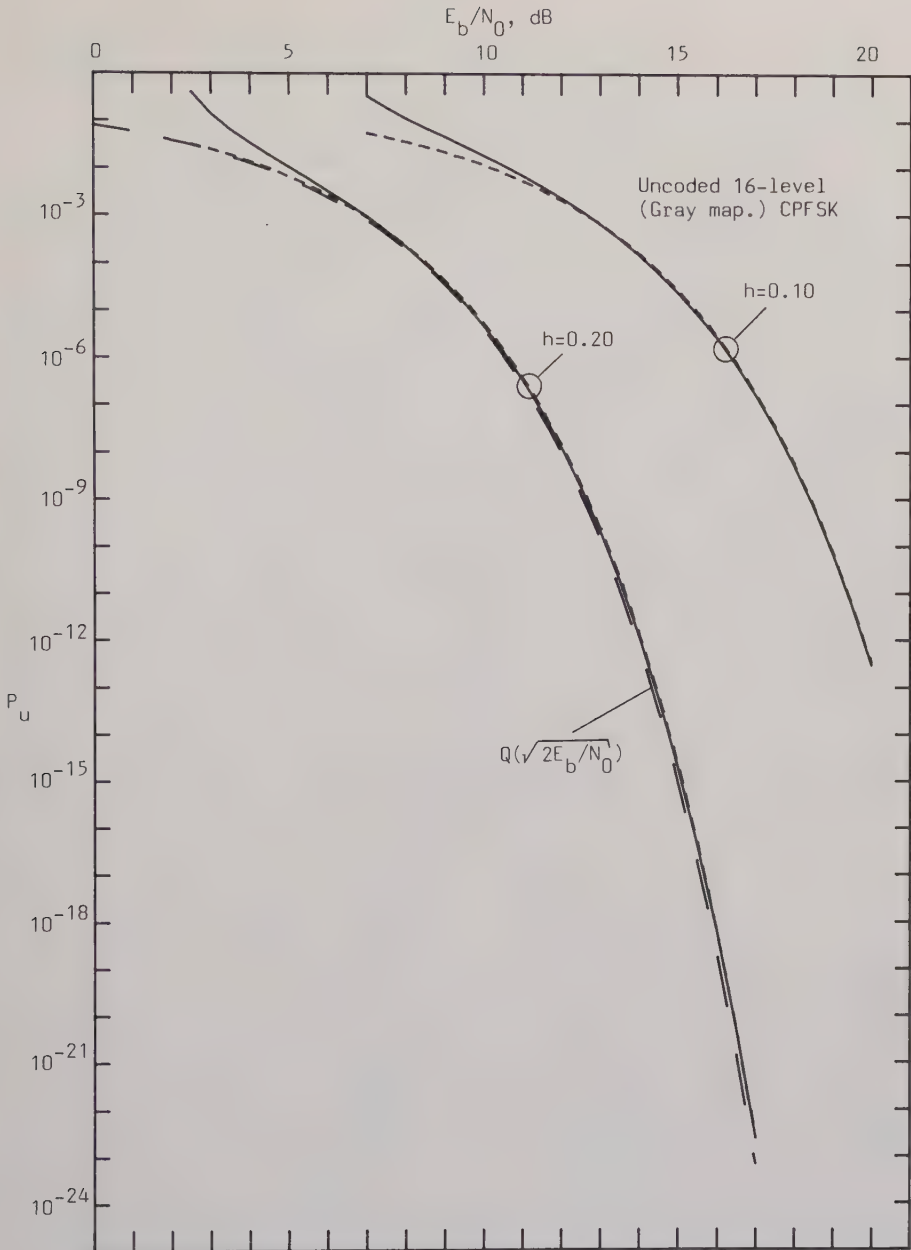


Figure 13. The upper bound (solid line) on the bit error probability for uncoded 16-level (Gray map.) CPFSK modulation. The dashed line is the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$.

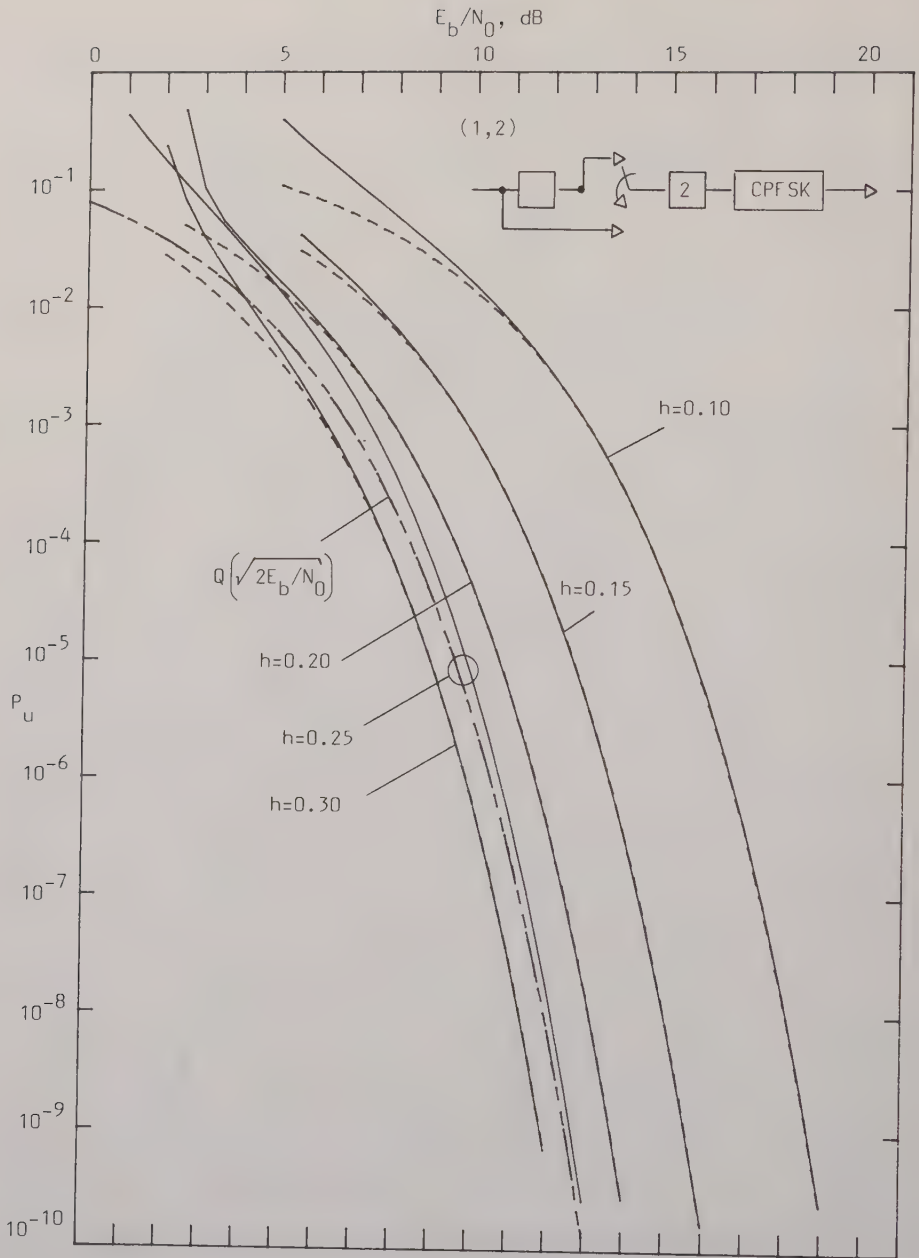


Figure 14. The upper bound P_u (solid line) on the bit error probability when the rate $1/2$ (1,2)-encoder is used with binary CPFSK modulation. The dashed line is the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$.

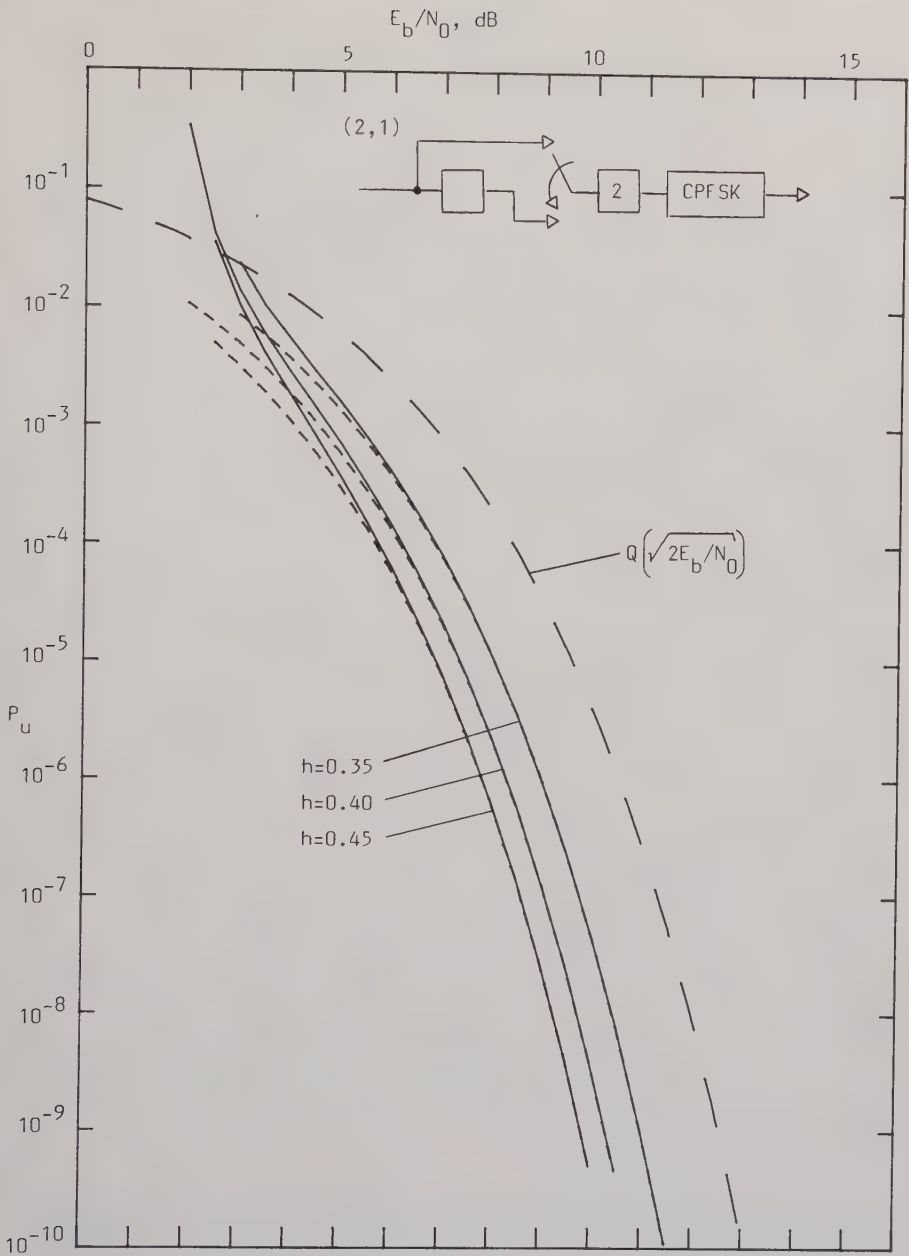


Figure 15. The upper bound P_u (solid line) on the bit error probability when the rate $1/2$ $(2,1)$ -encoder is used with binary CPFSK modulation. The dashed line is the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$.

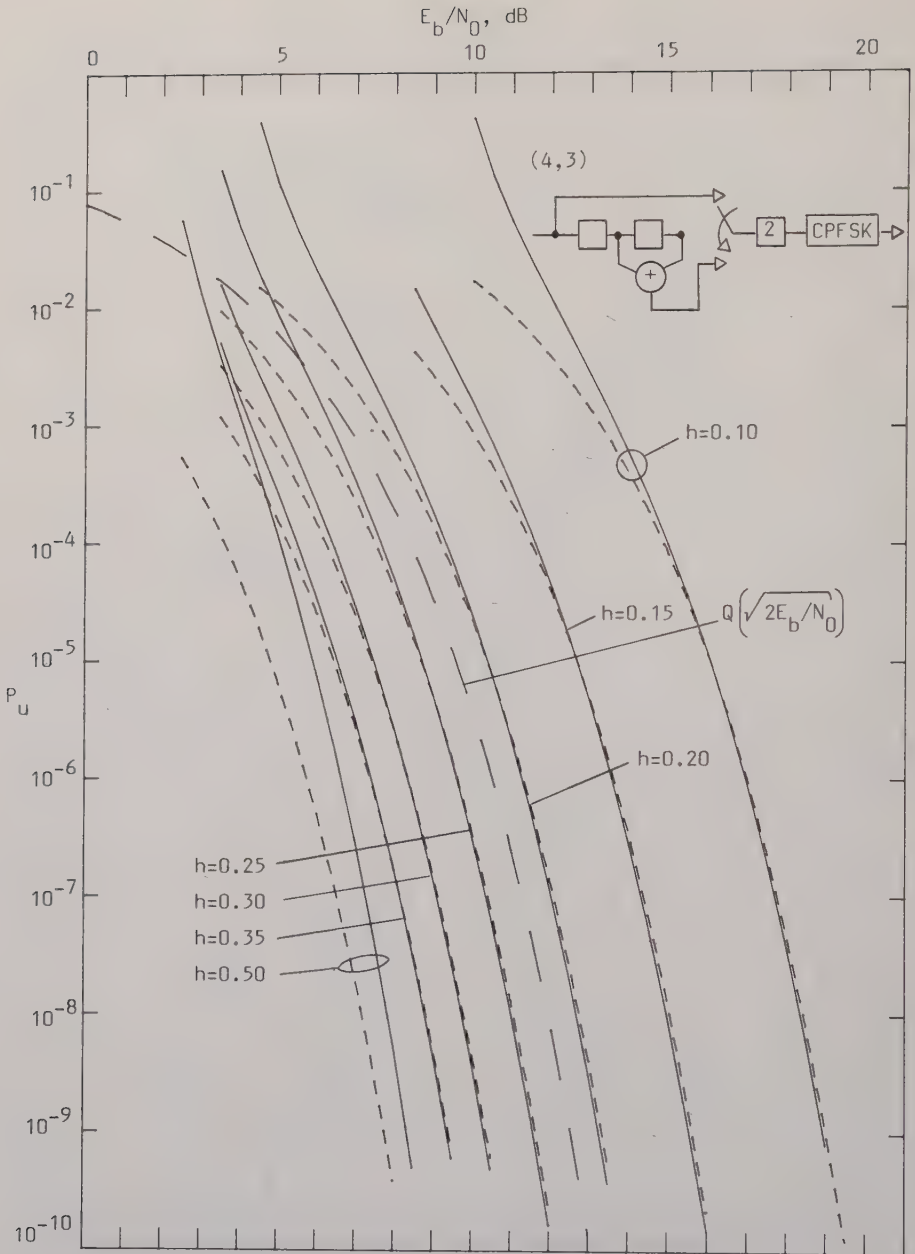


Figure 16. The upper bound P_u (solid line) on the bit error probability when the rate $1/2$ $(4,3)$ -encoder is used with binary CPFSK modulation. The dashed line is the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$. See also fig. 17.

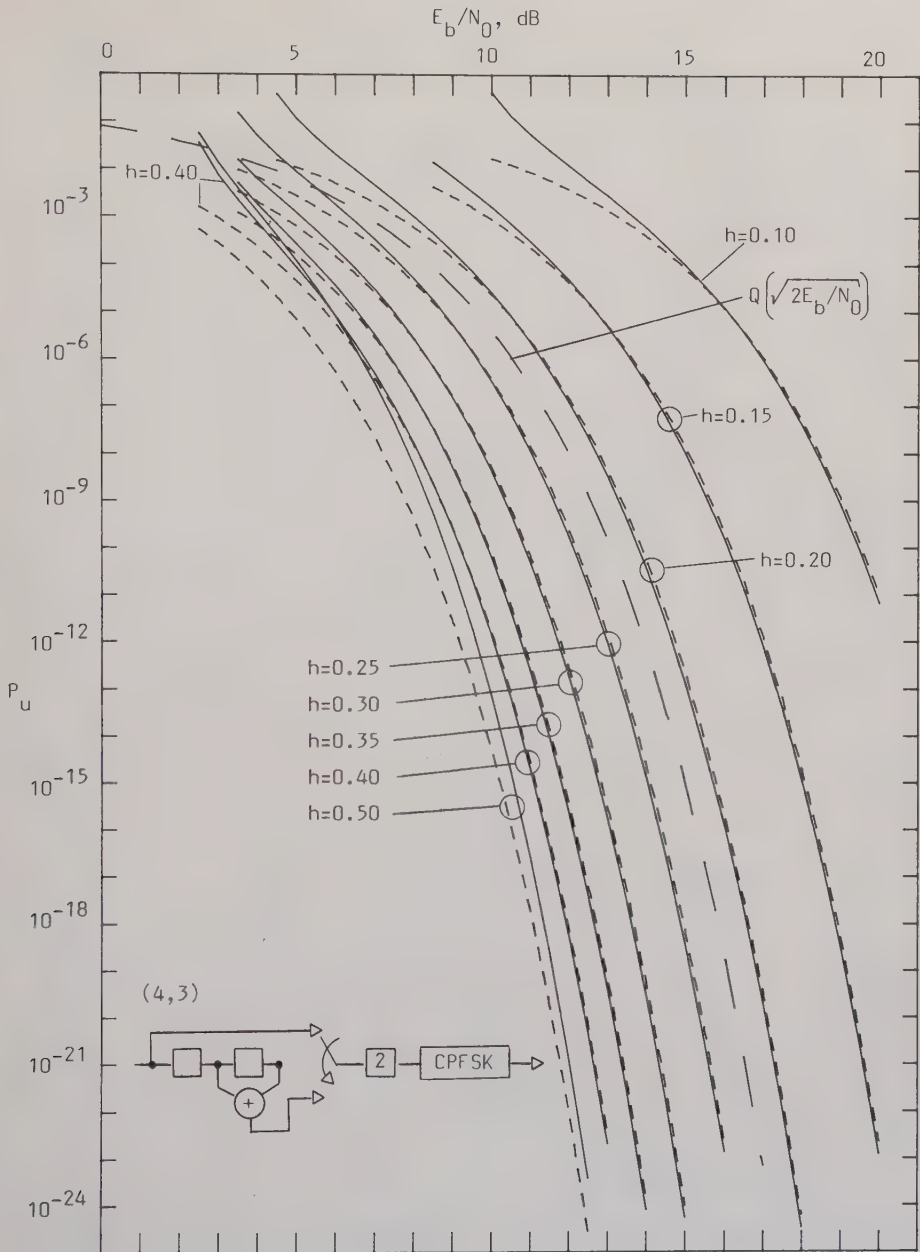


Figure 17. The upper bound P_u (solid line) on the bit error probability when the rate $1/2$ (4,3)-encoder is used with binary CPFSK modulation. The dashed line is the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$. See also fig. 16.

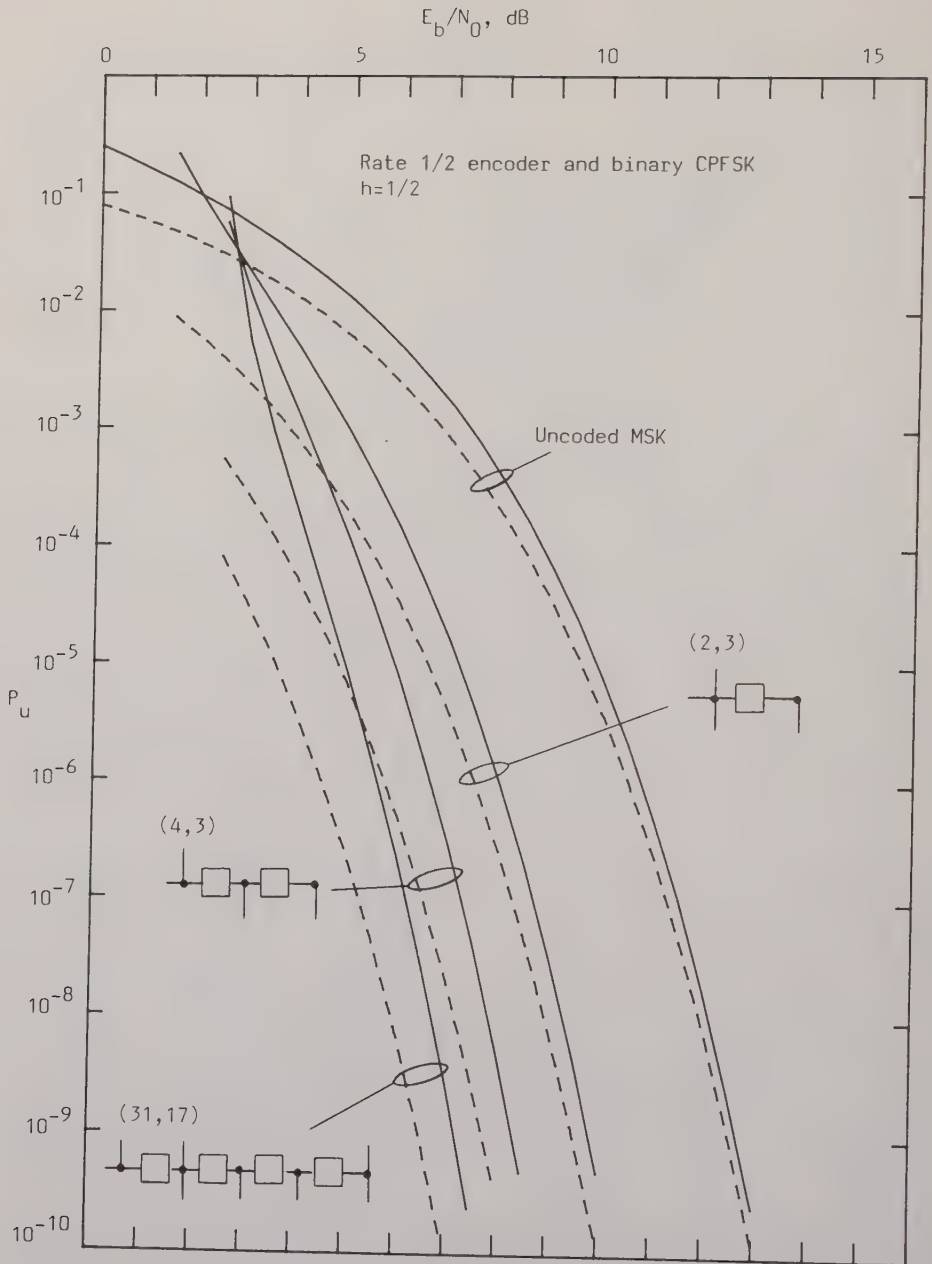


Figure 18. The upper bound P_u (solid line) on the bit error probability for 3 rate 1/2 encoded binary CPFSK schemes, $h=1/2$. The dashed line is the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$. See also fig. 19.

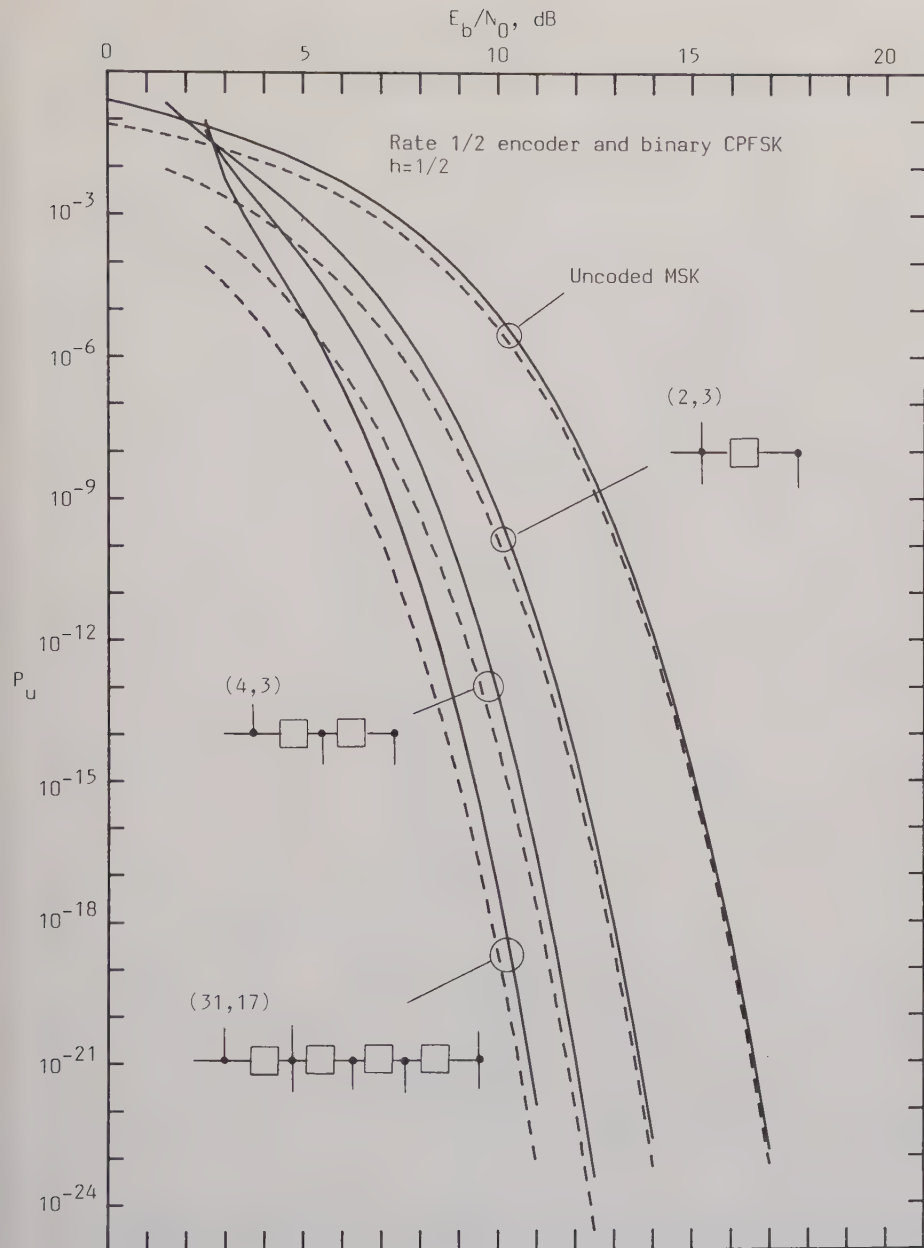


Figure 19. The upper bound P_u (solid line) on the bit error probability for 3 rate 1/2 encoded binary CPFSK schemes, $h=1/2$. The dashed line is the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$. See also fig. 18.

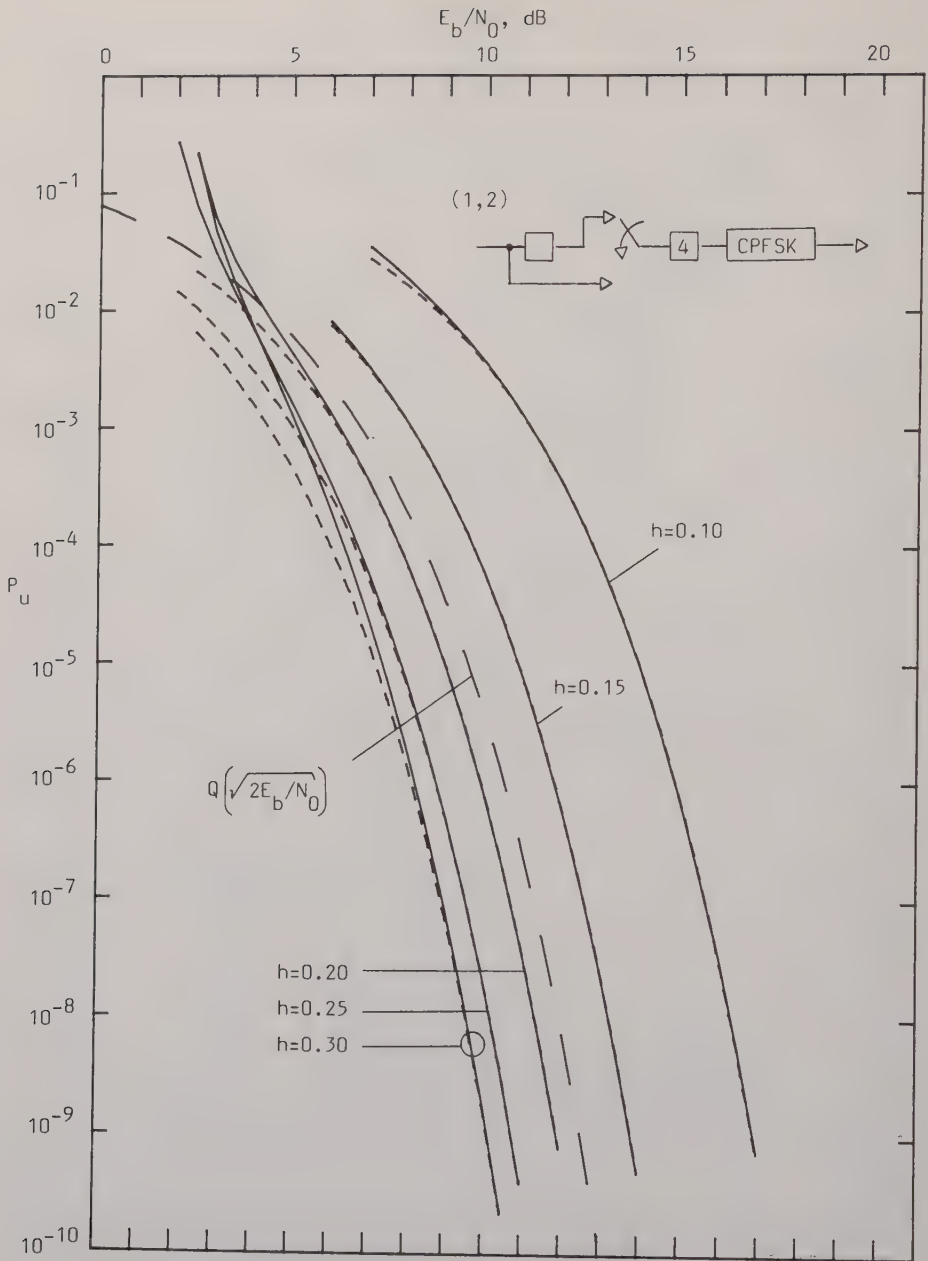


Figure 20. The upper bound P_u (solid line) on the bit error probability when the rate 1/2 (1,2)-encoder is used with 4-level (nat. bin.) CPFSK. The dashed line is the function $Q(\sqrt{2E_b/N_0})$.

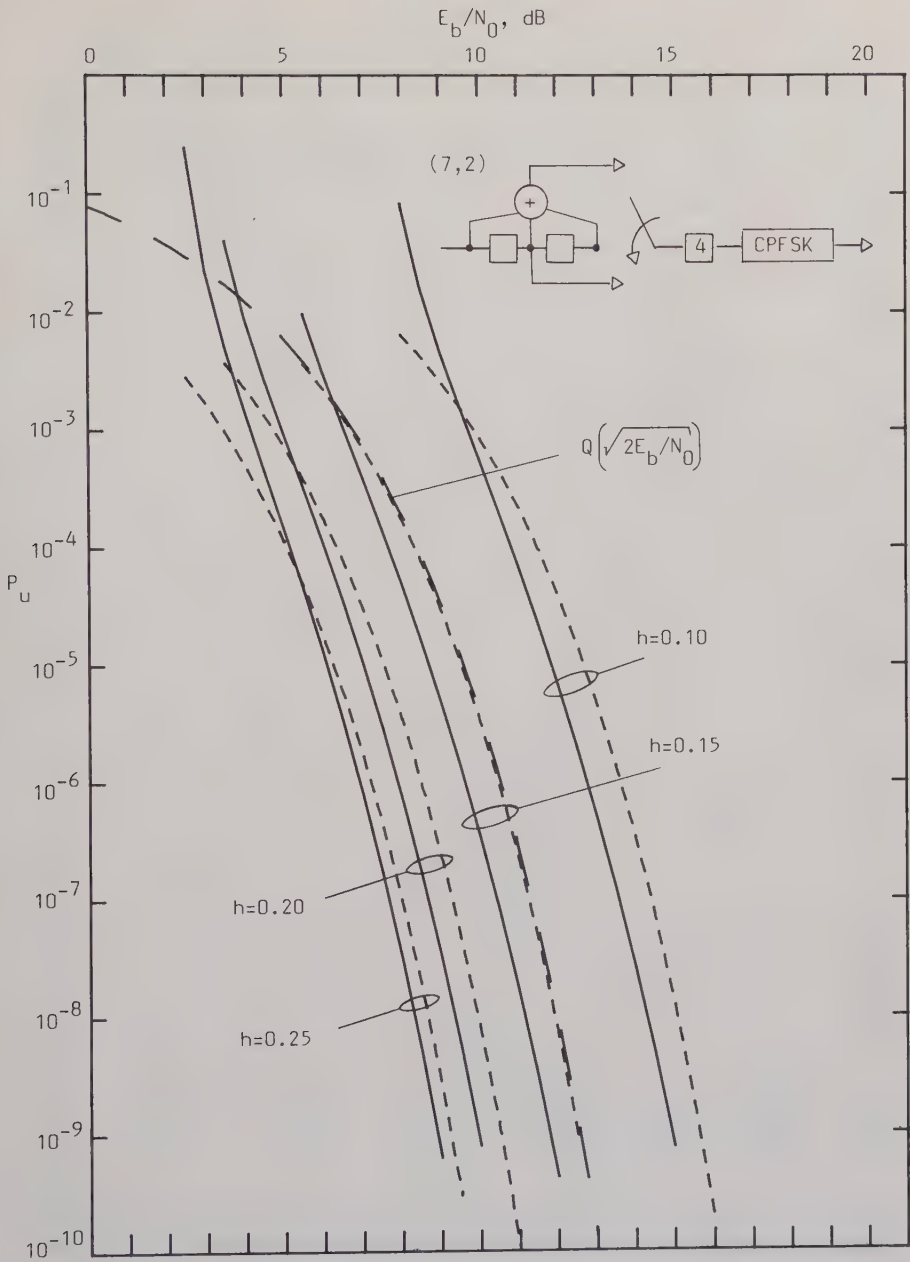


Figure 21. The upper bound P_u (solid line) on the bit error probability when the rate $1/2$ $(7,2)$ -encoder is used with 4-level (natural binary) CPFSK modulation. The dashed line is the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$.

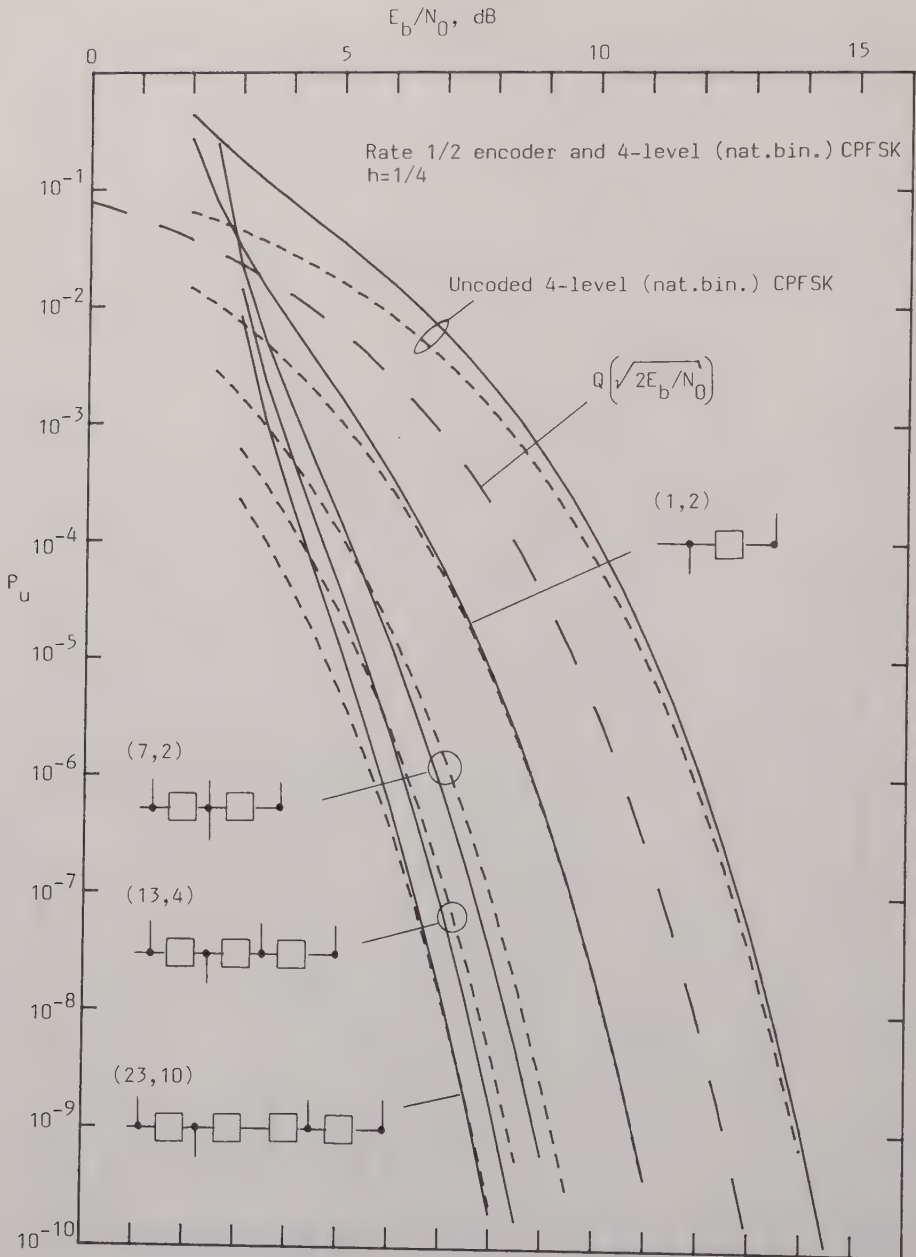


Figure 22. The upper bound P_u (solid line) on the bit error probability for 4 rate 1/2 encoded 4-level (nat.bin.) CPFSK schemes, $h=1/4$. The dashed line is the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$. See also fig. 23.

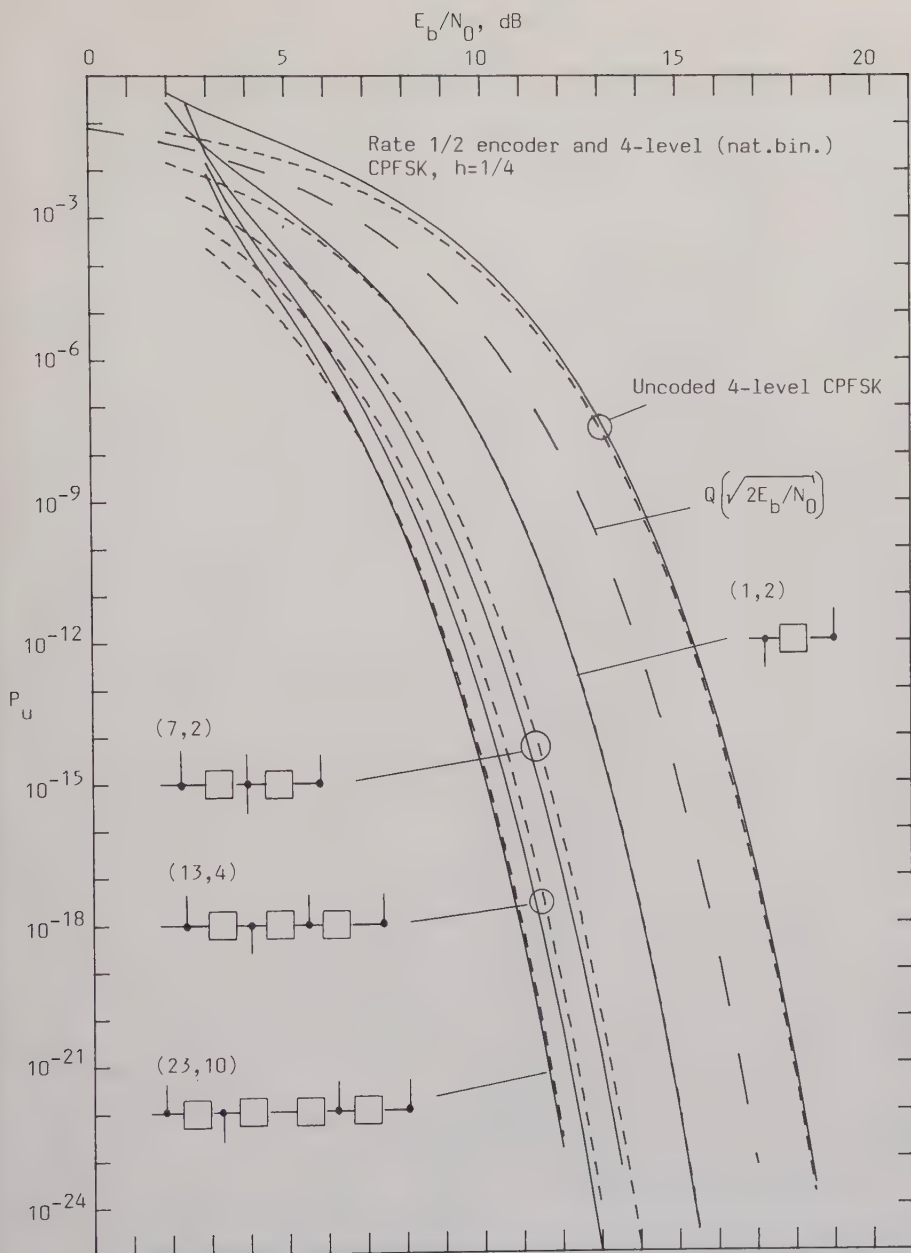


Figure 23. The upper bound P_u (solid line) on the bit error probability for 4 rate 1/2 encoded 4-level (nat.bin.) CPFSK schemes, $h=1/4$. The dashed line is the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$. See also fig. 22.

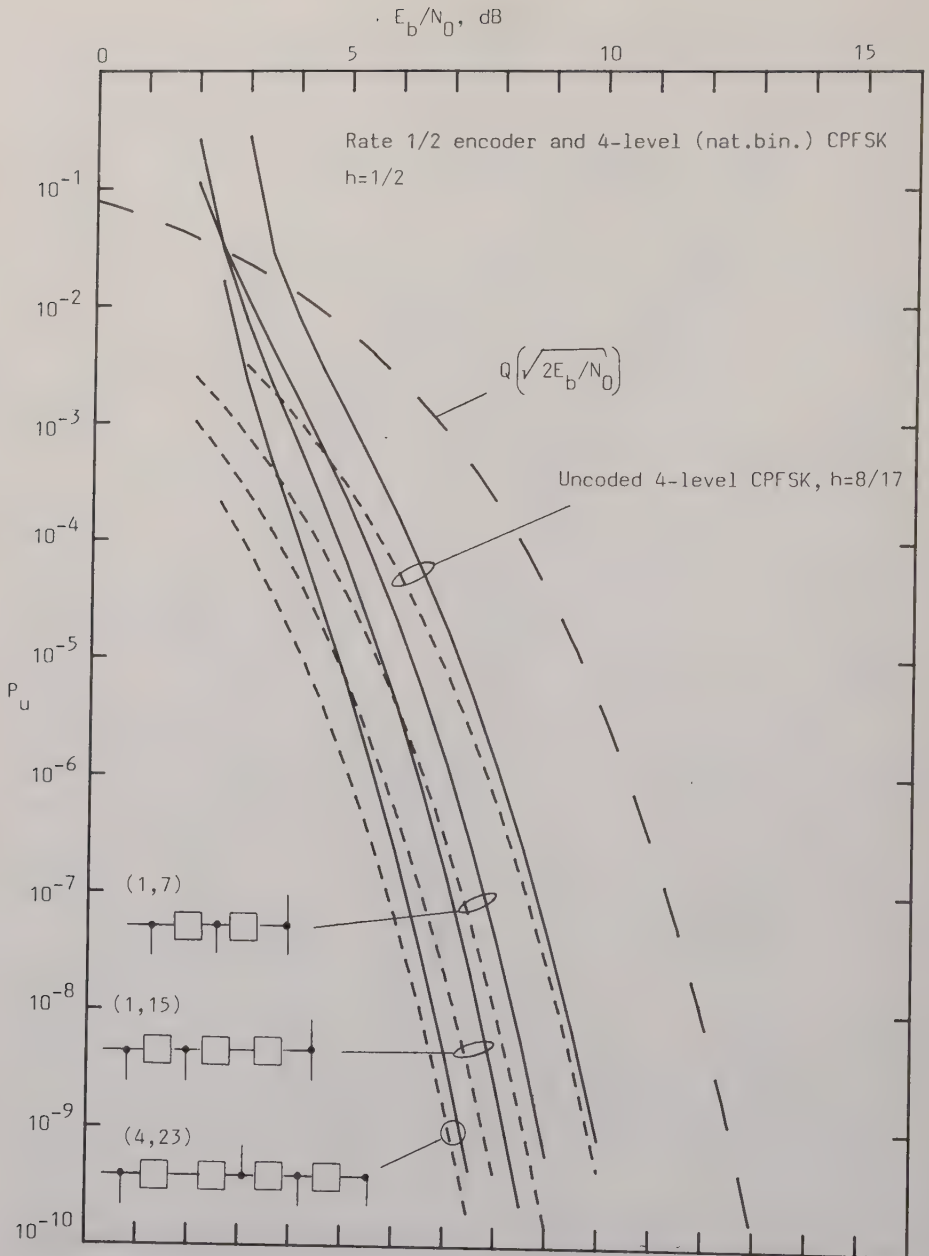


Figure 24. The upper bound P_u (solid line) on the bit error probability for 3 rate 1/2 encoded 4-level (natural binary) CPFSK schemes, $h=1/2$. The dashed line is the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$. See also fig. 25.

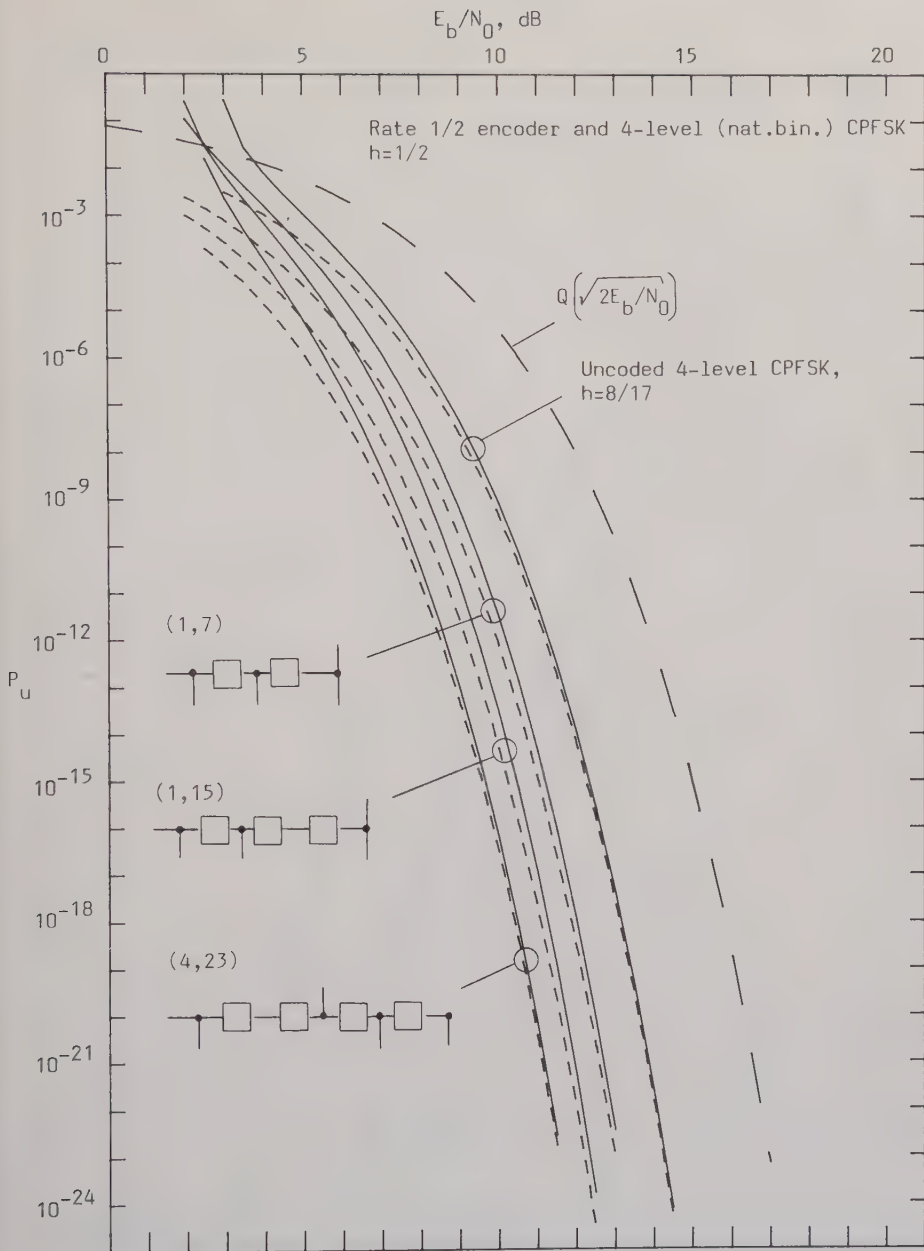


Figure 25. The upper bound P_u (solid line) on the bit error probability for 3 rate 1/2 encoded 4-level (nat.bin.) CPFSK schemes, $h=1/2$. The dashed line is the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$. See also fig. 24.

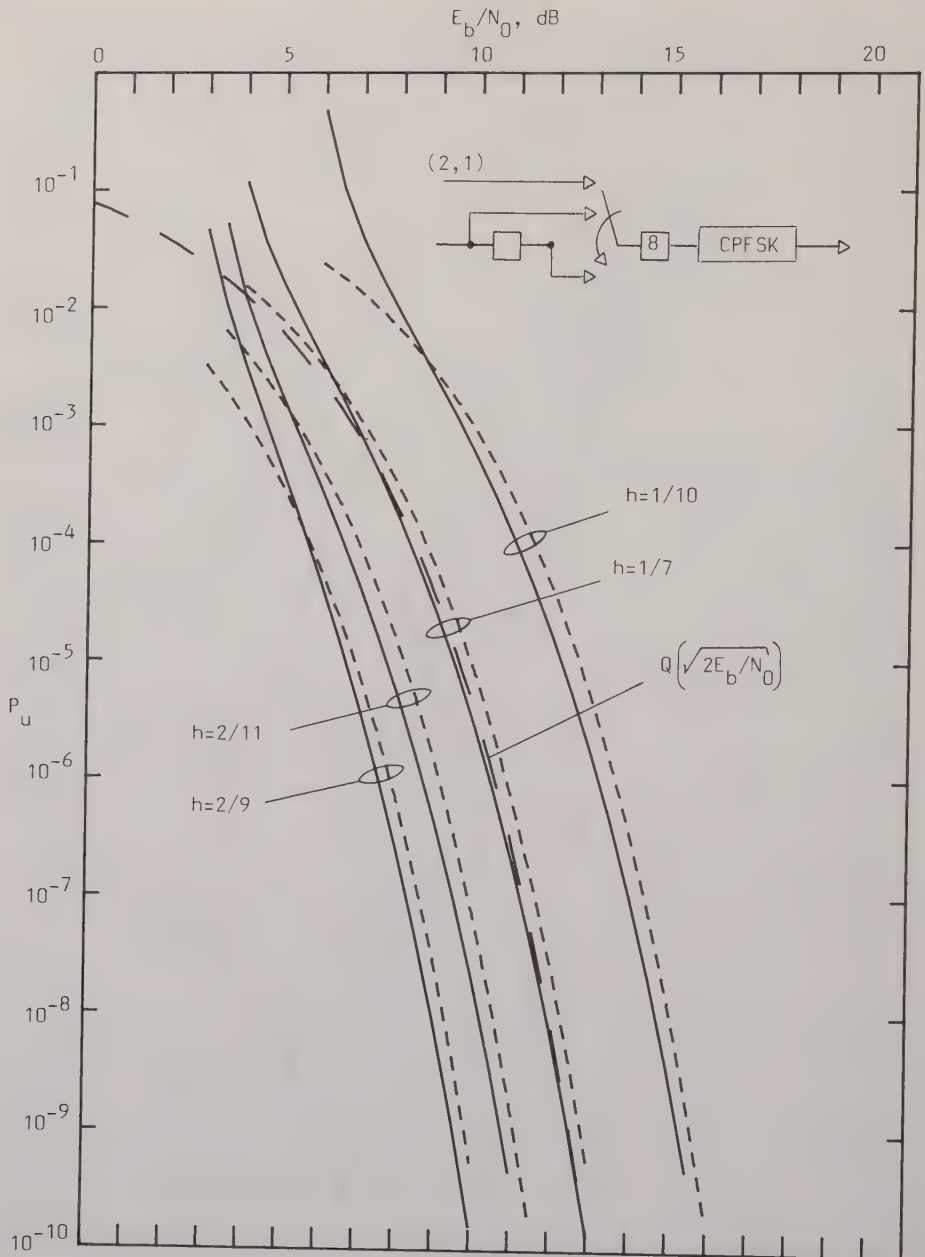


Figure 26. The upper bound P_u (solid line) on the bit error probability when the rate 2/3 (2,1)-encoder is used with 8-level (natural binary) CPFSK modulation. The dashed line is the function $Q\left(\sqrt{d_{\min}^2 E_b/N_0}\right)$.

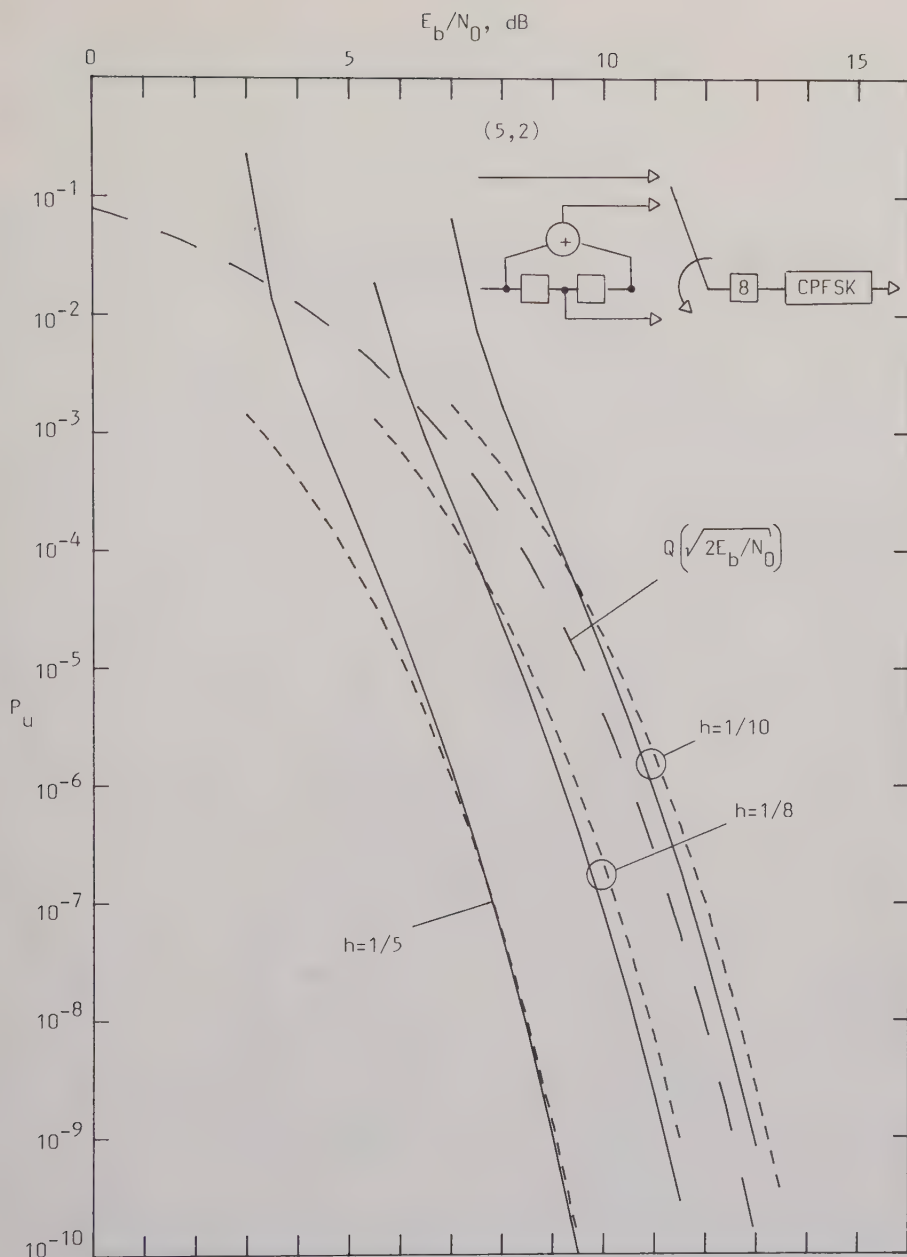


Figure 27. The upper bound P_u (solid line) on the bit error probability when the rate $2/3$ (5,2)-encoder is used with 8-level (nat.bin.) CPFSK modulation. The dashed line is the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$.

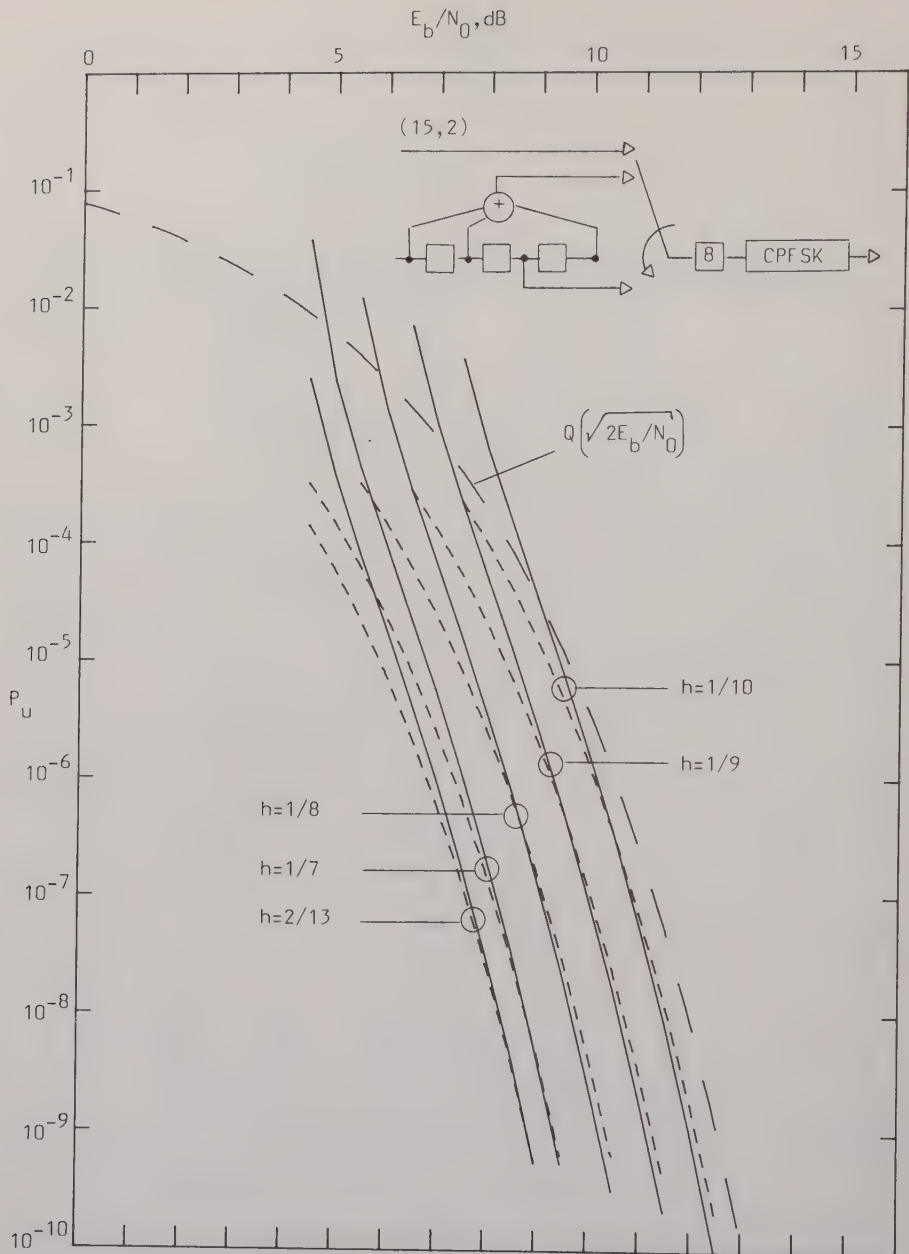


Figure 28. The upper bound P_u (solid line) on the bit error probability when the rate $2/3$ (15,2)-encoder is used with 8-level (nat.bin.) CPFSK modulation. The dashed line is the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$. See also fig. 29.

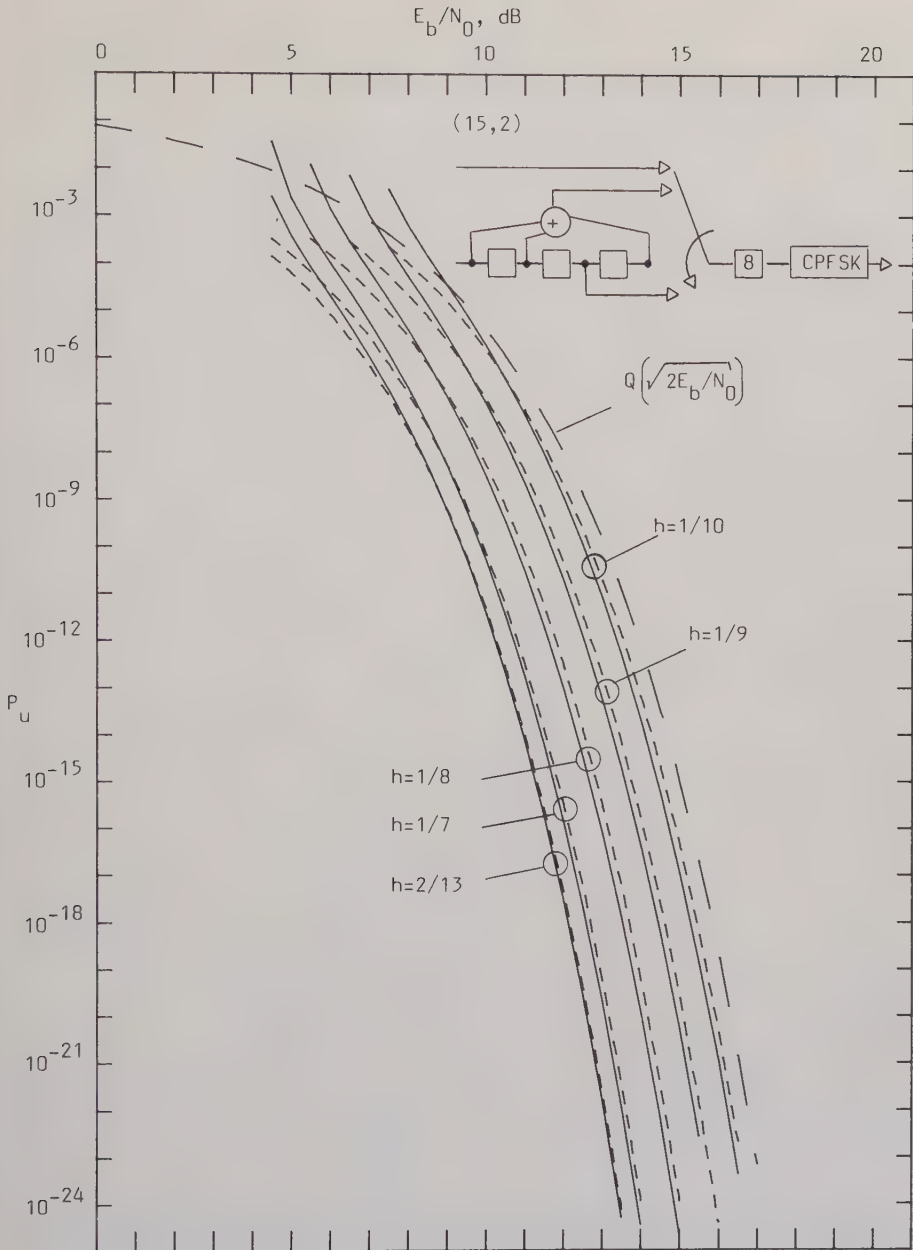


Figure 29. The upper bound P_u (solid line) on the bit error probability when the rate $2/3$ (15,2)-encoder is used with 8-level (nat.bin.) CPFSK modulation. The dashed line is the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$. See also fig. 28.

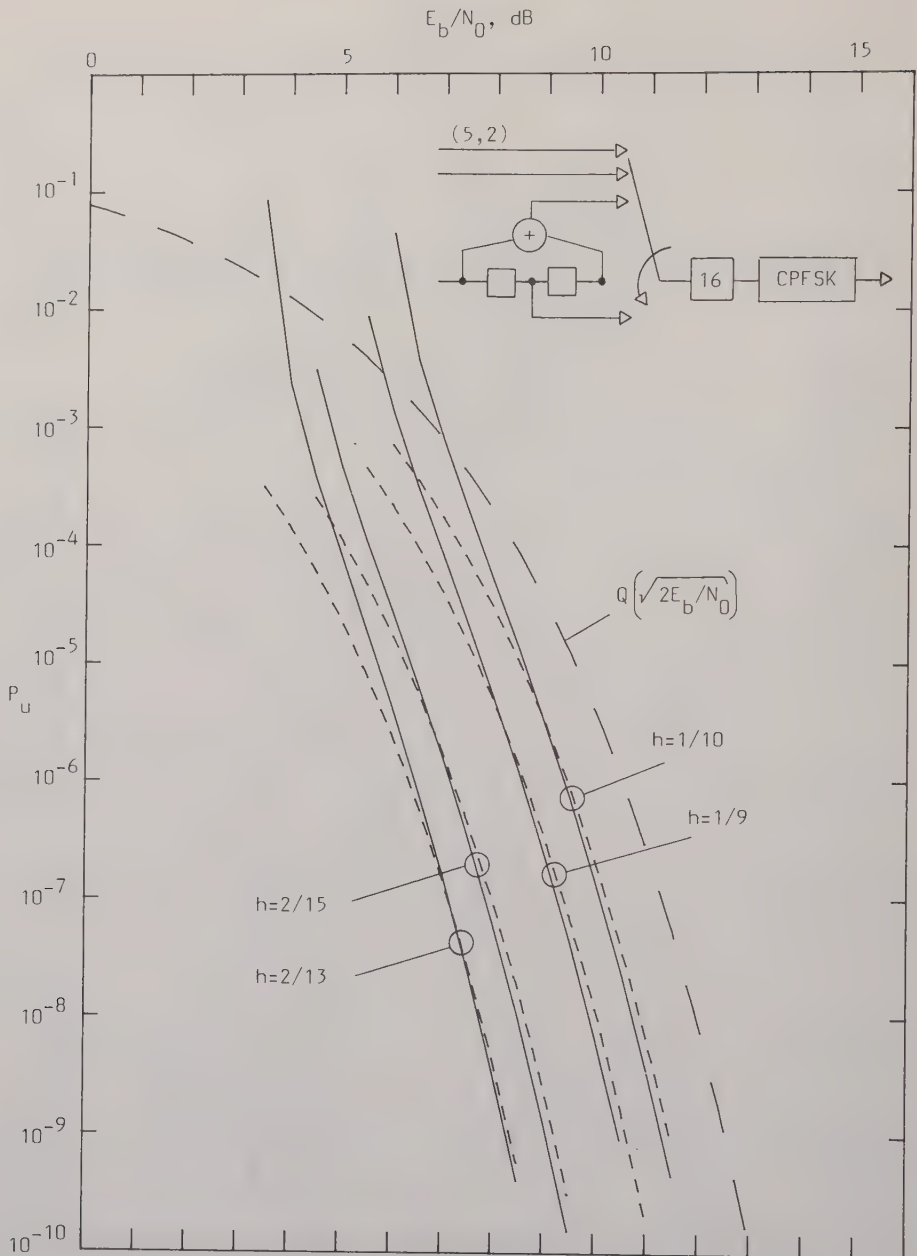


Figure 30. The upper bound P_u (solid line) on the bit error probability when the rate $3/4$ (5,2)-encoder is used with 16-level (nat.bin.) CPFSK modulation. The dashed line is the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$. See also fig. 31.

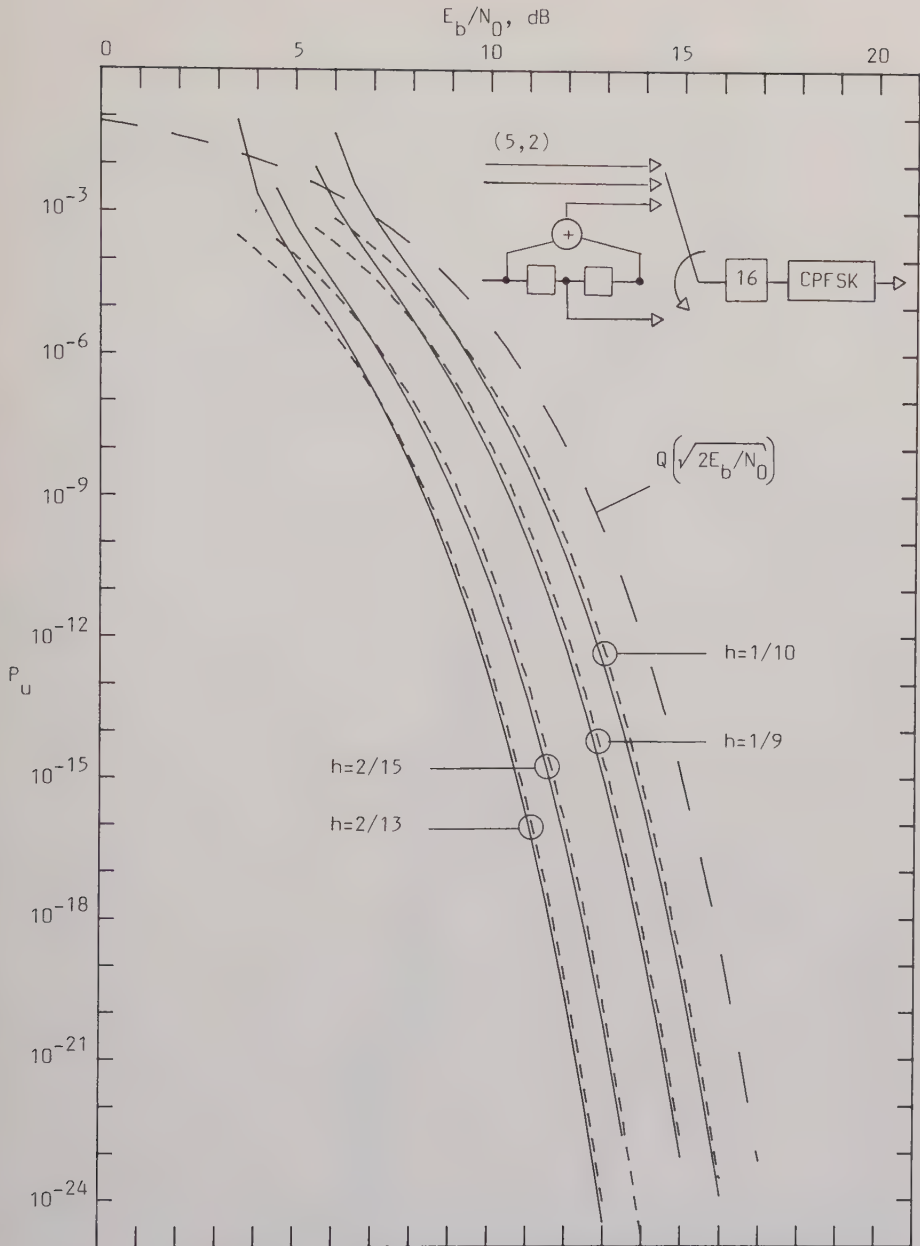


Figure 31. The upper bound P_u (solid line) on the bit error probability when the rate $3/4$ $(5,2)$ -encoder is used with 16-level (nat.bin.) CPFSK modulation. The dashed line is the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$. See also fig. 30.

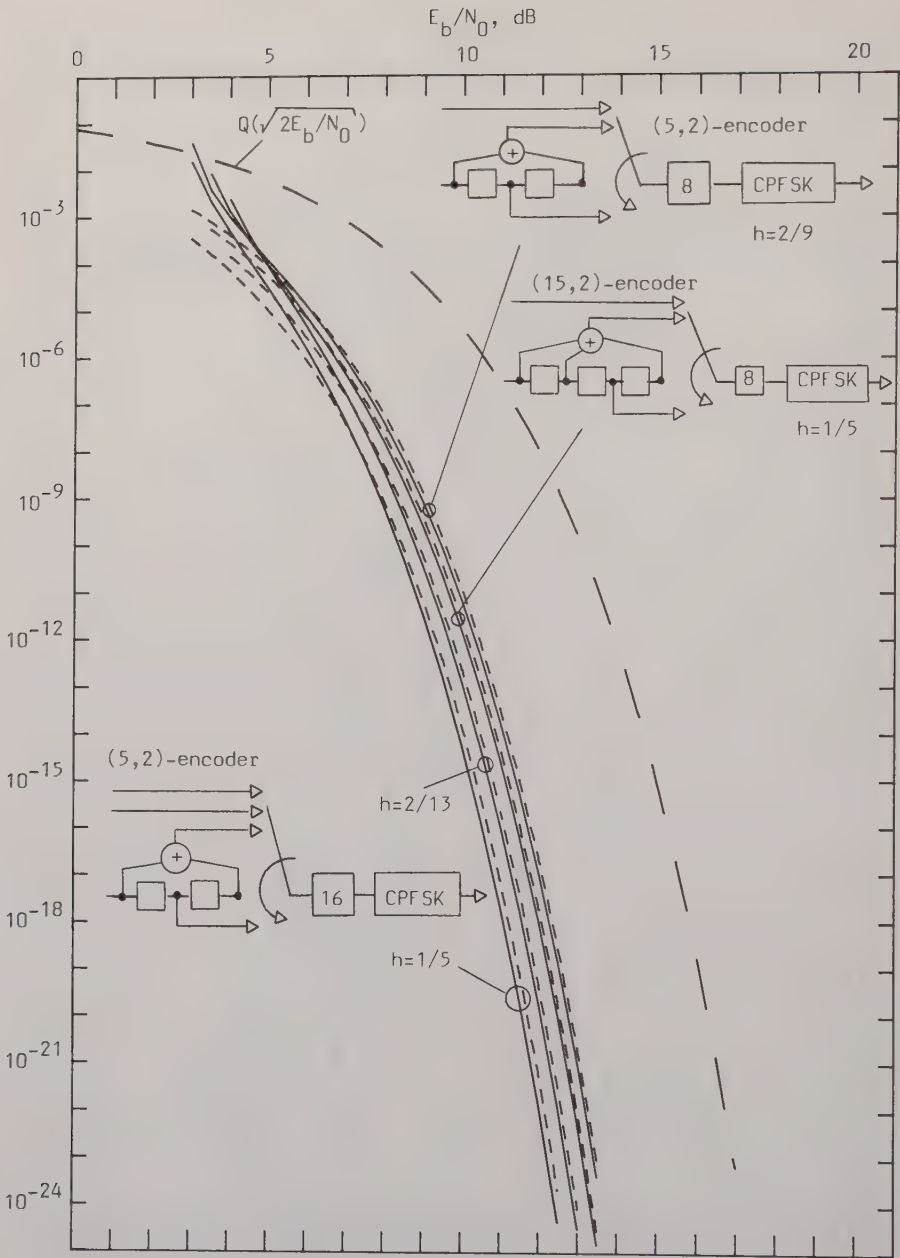


Figure 32. The upper bound P_U (solid line) on the bit error probability. All schemes in this figure have reached the encoder-independent upper bound on $d_{\min}^2$ considered in Part III. The dashed line is the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$.

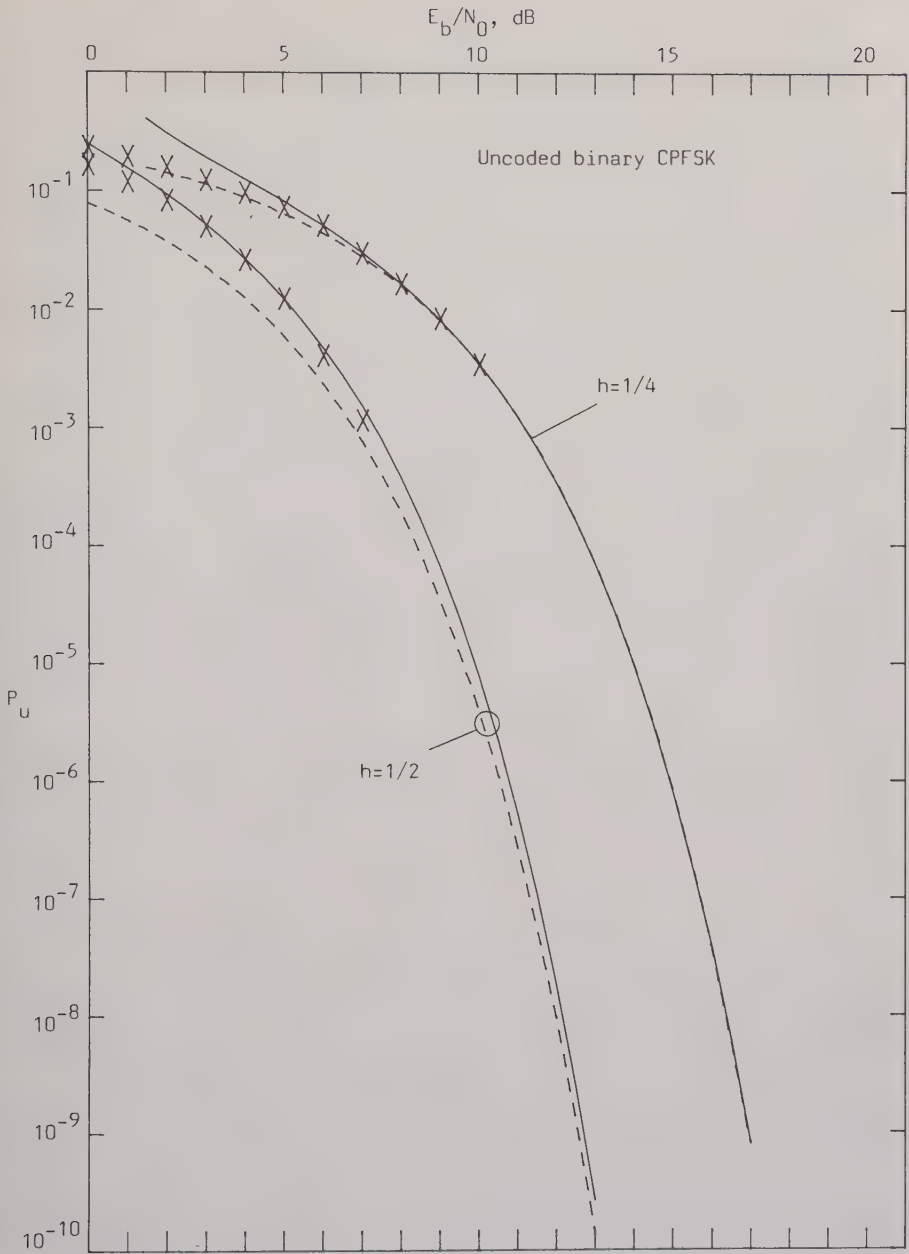


Figure 33. Simulated bit error probabilities (x) for uncoded binary CPFSK modulation. The upper bound P_u (solid line) and the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$ (dashed line) are also given in this figure.

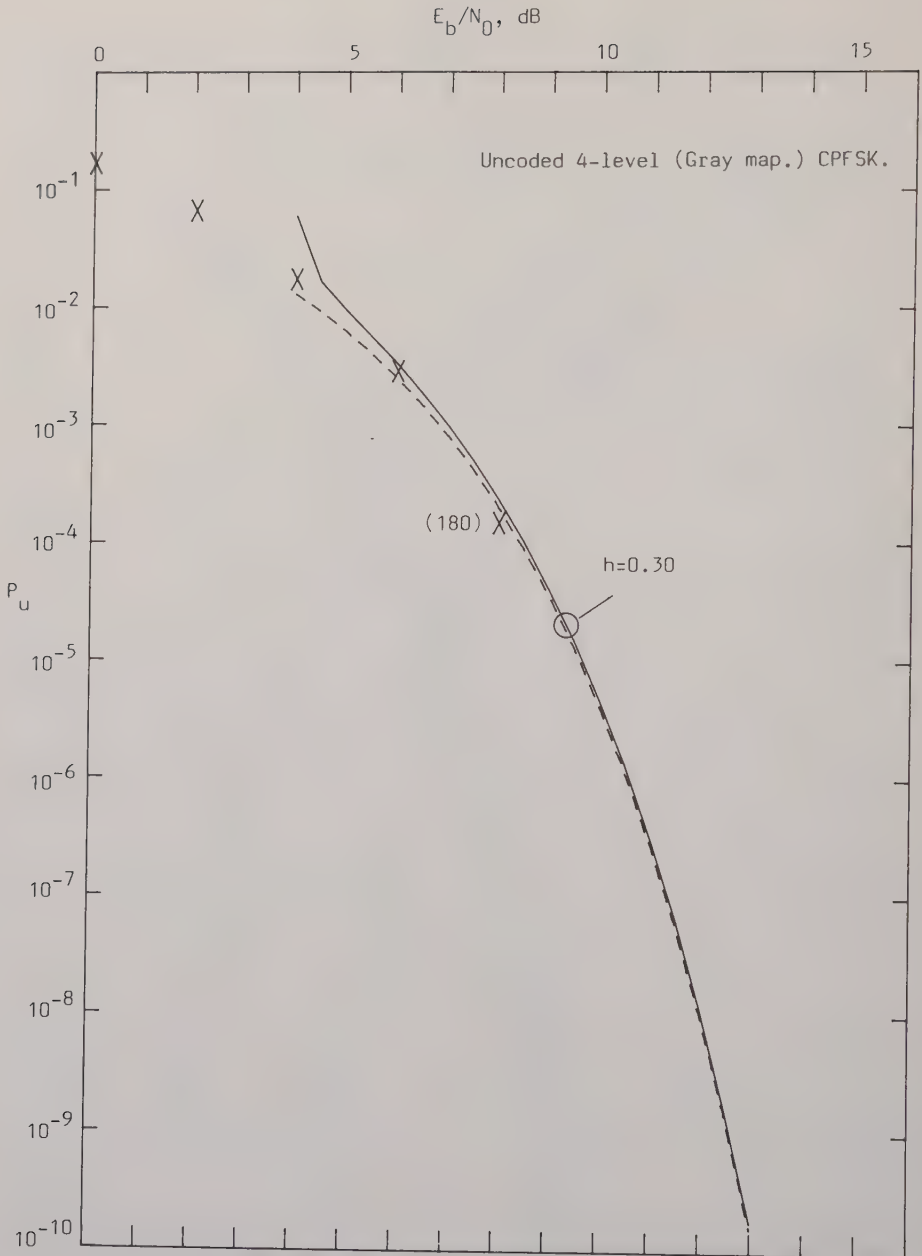


Figure 34. Simulated bit error probabilities (x) for uncoded 4-level (Gray map.) CPFSK modulation. The upper bound P_u (solid line) and the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$ (dashed line) are also given in this figure.

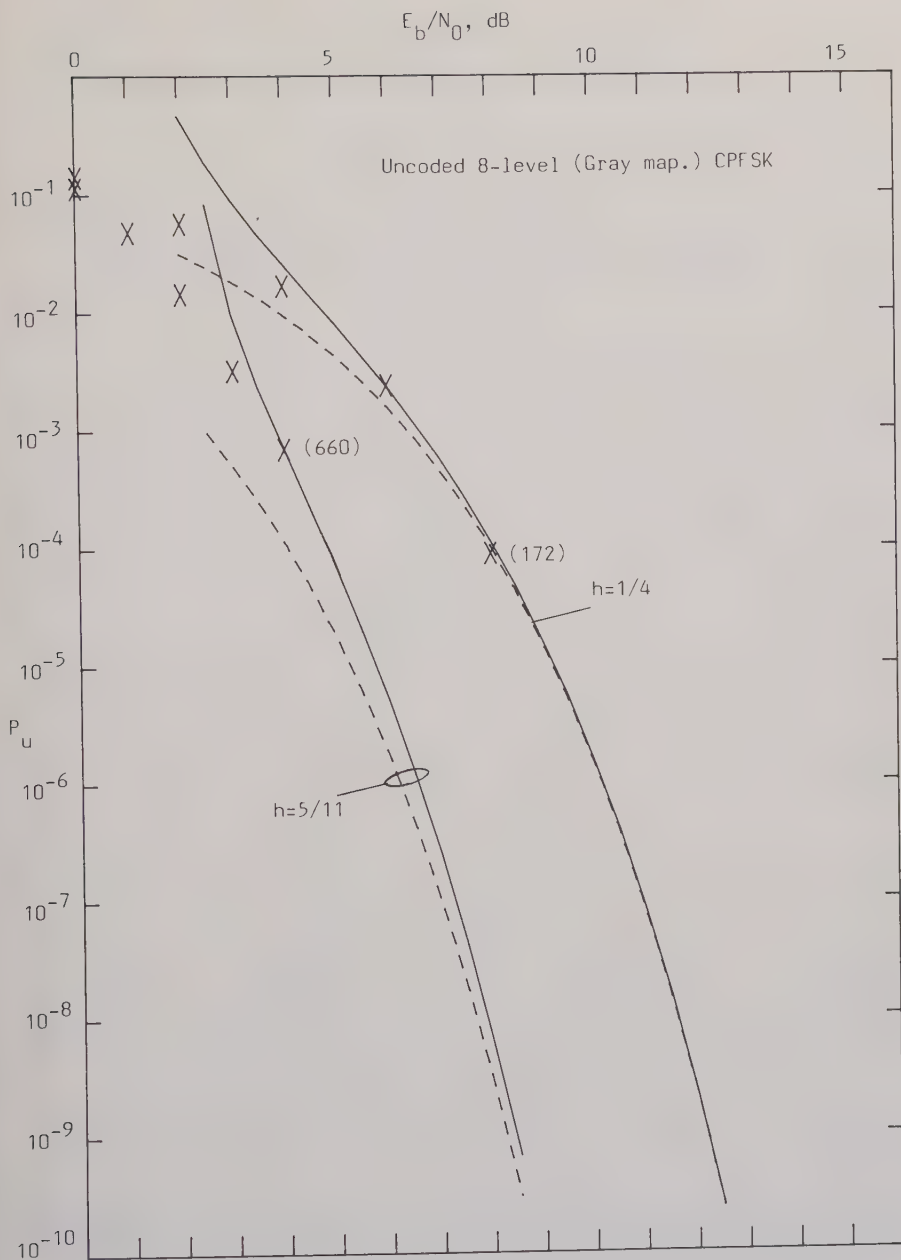


Figure 35. Simulated bit error probabilities (x) for uncoded 8-level (Gray map.) CPFSK modulation. The upper bound P_u (solid line) and the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$ (dashed line) are also given in this figure.

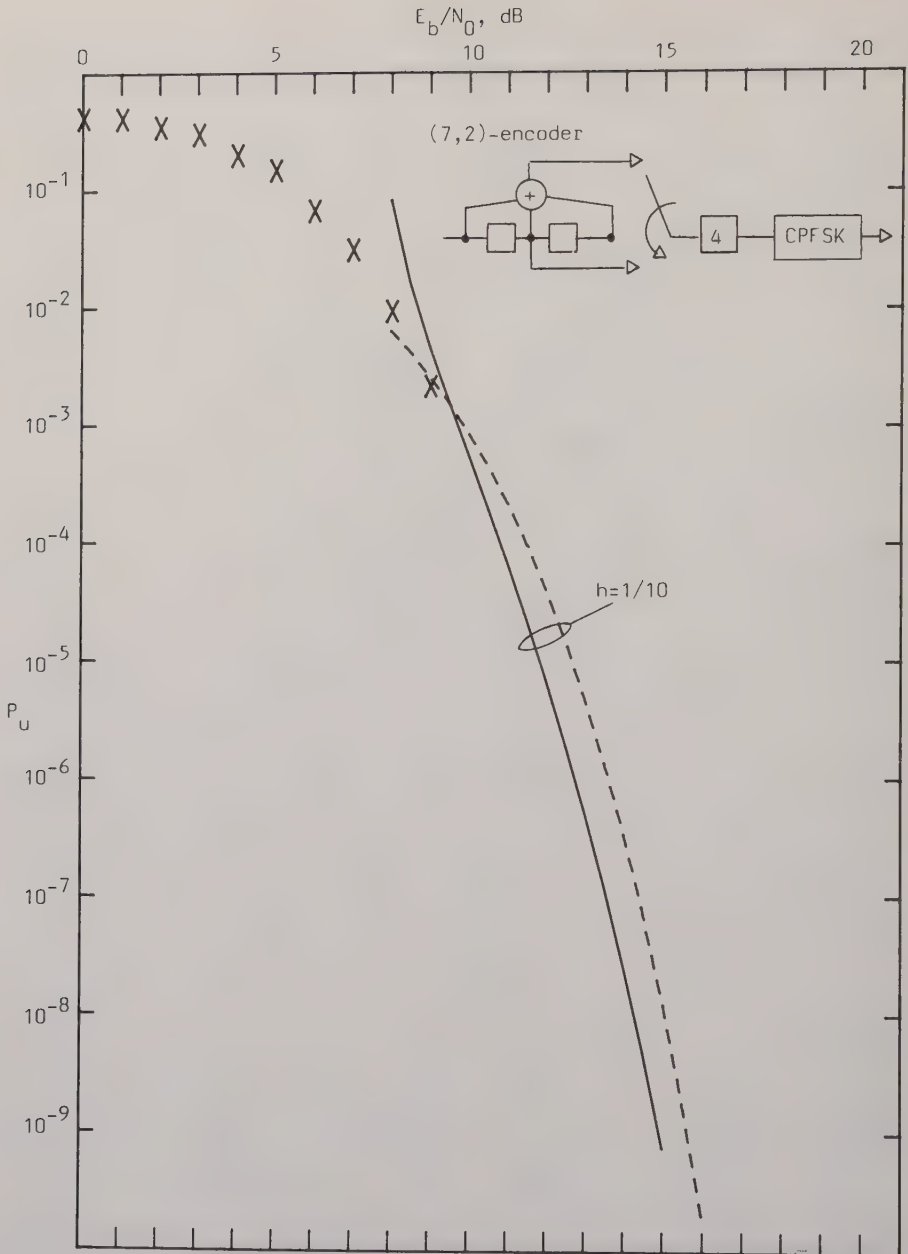


Figure 36. Simulated bit error probabilities (x) when the rate $1/2$ (7,2)-encoder is used with 4-level (nat.bin.) CPFSK modulation. The upper bound P_u (solid line) and the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$ (dashed line) are also given in this figure.

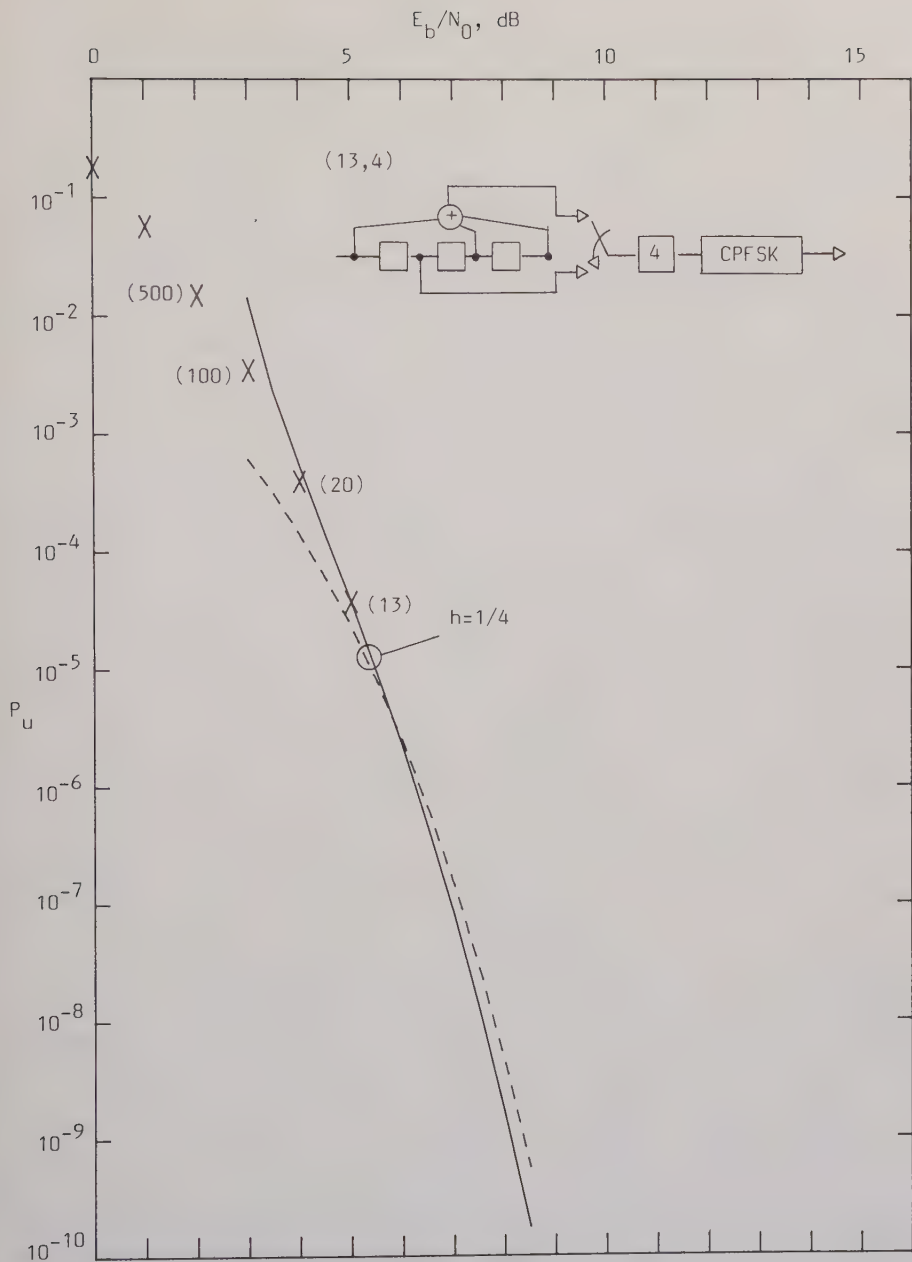


Figure 37. Simulated bit error probabilities (x) when the rate $1/2$ (13,4)-encoder is used with 4-level (nat,bin.) CPFSK modulation. The upper bound P_u (solid line) and the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$ (dashed line) are also given in this figure.

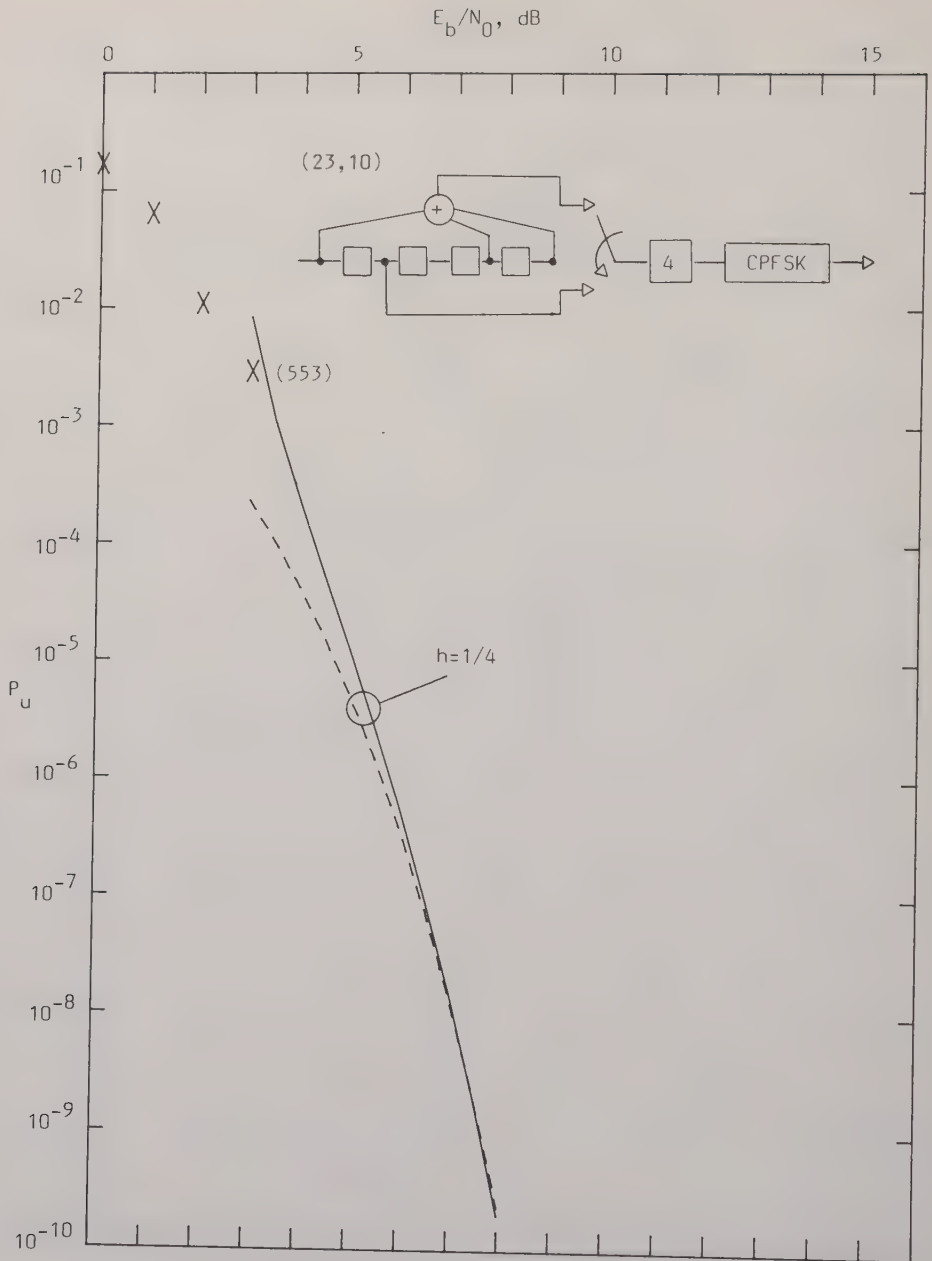


Figure 38. Simulated bit error probabilities (x) when the rate 1/2 (23,10)-encoder is used with 4-level (nat.bin.) CPFSK modulation. The upper bound P_u (solid line) and the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$ (dashed line) are also given in this figure.

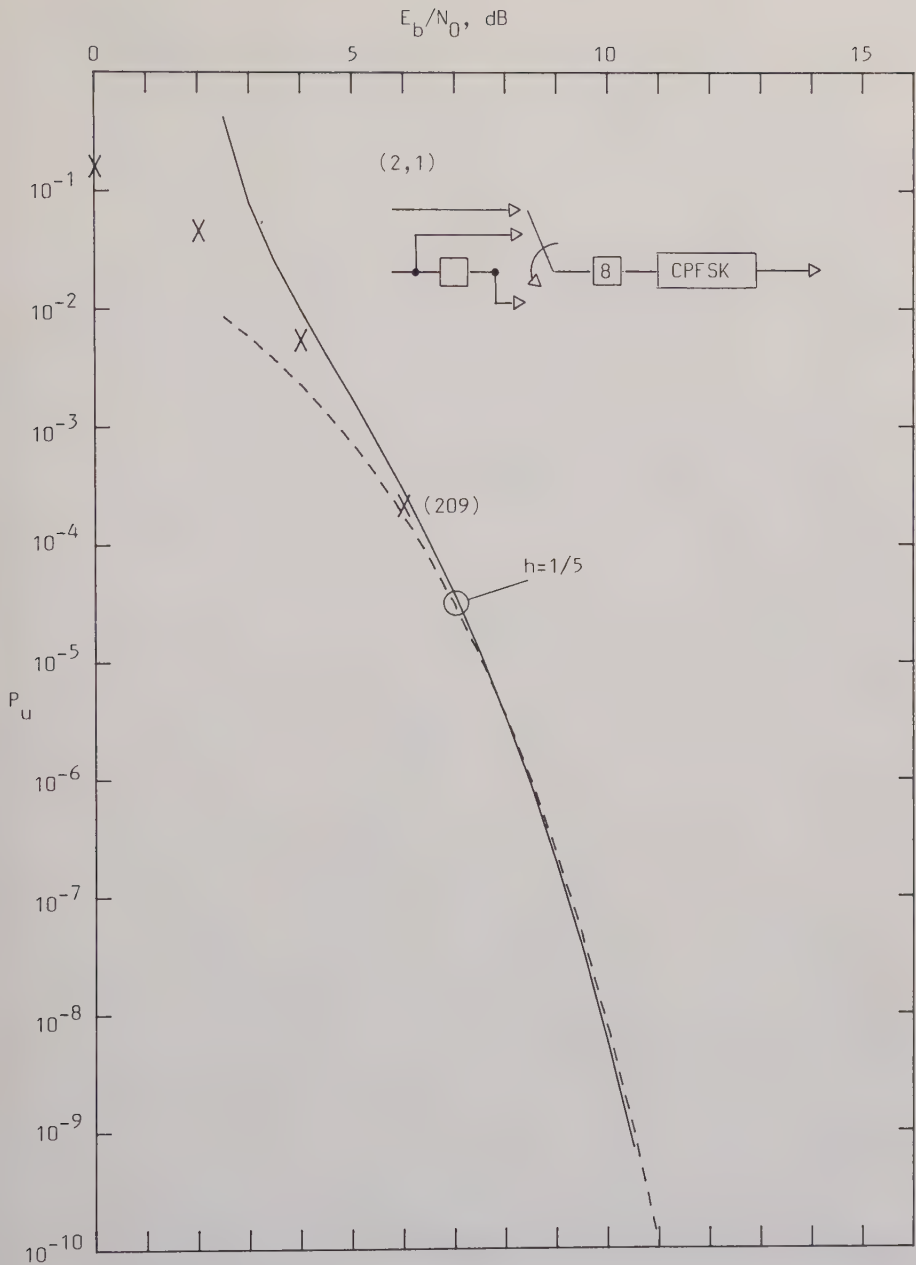


Figure 39. Simulated bit error probabilities (x) when the rate $2/3$ (2,1)-encoder is used with 8-level (nat.bin.) CPFSK modulation. The upper bound P_u (solid line) and the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$ (dashed line) are also given in this figure.

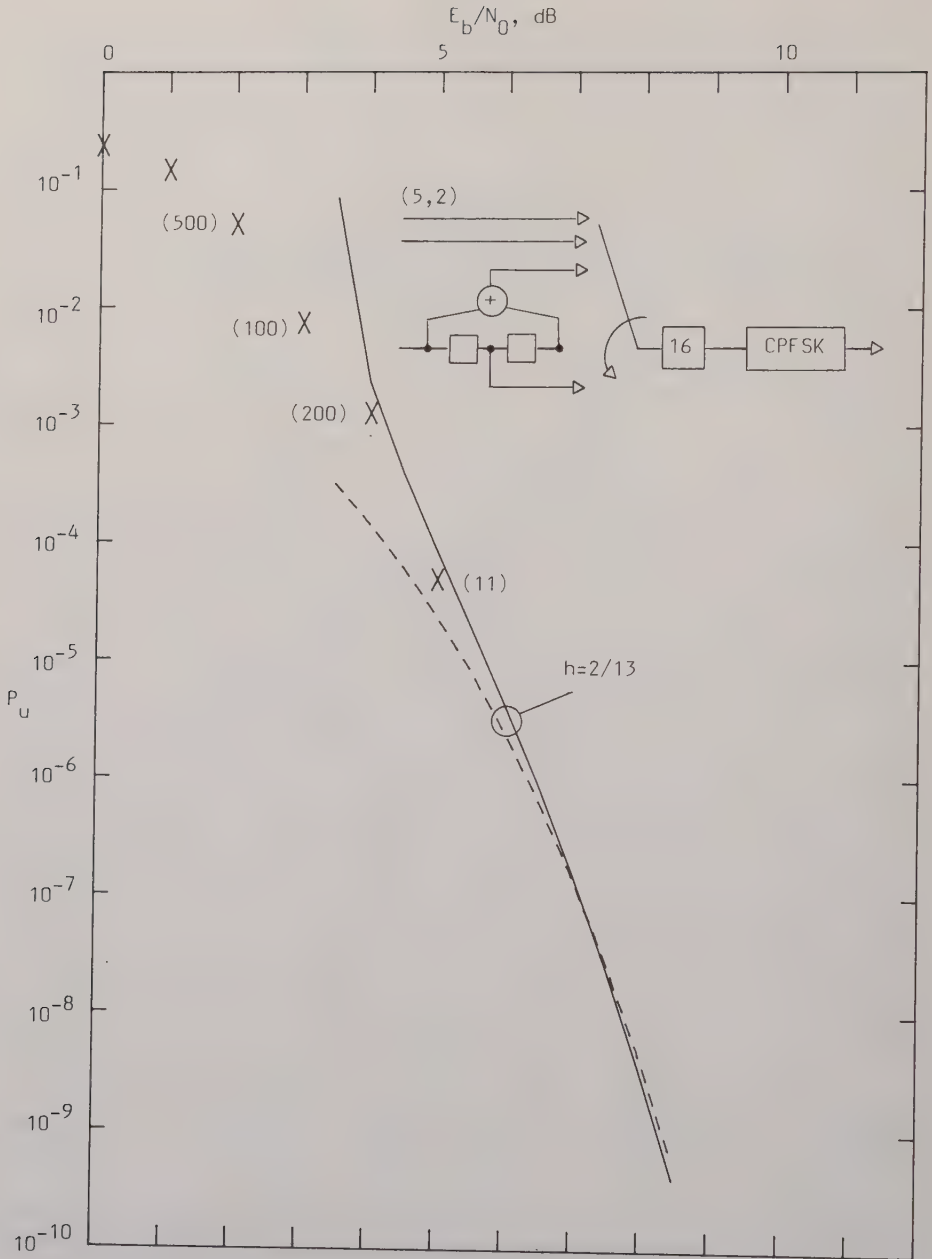


Figure 40. Simulated bit error probabilities (x) when the rate 3/4 (5,2)-encoder is used with 16-level (nat.bin.) CPFSK modulation. The upper bound P_u (solid line) and the function $Q(\sqrt{d_{\min}^2 E_b/N_0})$ (dashed line) are also given in this figure.

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